Some properties of critical Dirac equations

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In this talk I will present a classification result for critical Dirac equations on the Euclidean space, appearing naturally in conformal spin geometry and in variational problems related to critical Dirac equations on spin manifolds, see e.g. [1, 6]. Moreover, two-dimensional critical Dirac equations recently attracted a considerable attention as effective equations for wave propagation in honeycomb structures, see e.g. [2, 5]. Exploiting the conformal invariance, ground states can be classified [4]. This is the spinorial counterpart of the well-known result for the Yamabe equation.

Time permitting, I will also present similar results obtained for conformal Dirac-Einstein equations, consisting of a Dirac equation coupled with a Laplace-type equation [3].

Joint work with Ali Maalaoui (Clark University, USA), Andrea Malchiodi (SNS, Pisa) and Ruijun Wu (SISSA, Trieste).

References

[1] B. Ammann, J.-F. Grosjean, E. Humbert, and B. Morel, *A spinorial analogue of Aubin's inequality*, Math. Z., 260 (2008), pp. 127–151.

[2] W. Borrelli, R.L.Frank, *Sharp decay estimates for critical Dirac equations*, Trans. Amer. Math. Soc., 373 (2020), 2045-2070.

[3] W. Borrelli, A. Maalaoui, *Some Properties of Dirac–Einstein Bubbles*, J. Geom. Anal., 31, 5766-5782 (2021).

[4] W. Borrelli, A. Malchiodi, R. Wu, *Ground state Dirac bubbles and Killing spinors*, Commun. Math. Phys., 383, 1151-1180 (2021).

[5] C. L. Fefferman and M. I. Weinstein, *Waves in honeycomb structures*, Journées équations aux dérivées partielles, Talk no. 12. https://doi.org/10.5802/jedp.95 (2012).

[6] T. Isobe, *Nonlinear Dirac equations with critical nonlinearities on compact Spin manifolds*, J. Funct. Anal., 260 (2011), pp. 253–307.

Solutions for the fractional Yamabe problem with singularities: existence, construction and qualitative properties.

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The so called fractional Yamabe problem in Conformal Geometry consists in finding a metric conformal to a given one and which has constant fractional curvature. From the analytic point of view, this problem becomes a non-local semilinear elliptic PDE with critical (for the Sobolev embedding) power nonlinearity.

In this talk, we will focus on the Euclidean space, allowing the presence of singularities. We will show existence of solutions under necessary assumptions on the singular set, how to construct these solutions and we will also provide qualitative properties they always satisfy.

I will show the main ideas to develop the results contained in different works done in collaborations with several people.

References

[1] W. Ao, A. DelaTorre, M. d. M. González and J. Wei. A gluing approach for the fractional Yamabe problem with prescribed isolated singularities. Journal für die reine und angewandte Mathematik (Crelle journal), 2020 (763), 25-78.

[2] W. Ao, H. Chan, A. DelaTorre, M. A. Fontelos, M. d. M. González, J. Wei. On higher-dimensional singularities for the fractional Yamabe problem: A nonlocal Mazzeo–Pacard program. Duke Math Journal Volume 168, Number 17 (2019), 3297-3411.

[3] W. Ao, H. Chan, A. DelaTorre, M. A. Fontelos, M. d. M. Gonzáez, J. Wei. ODE-methods in non-local equations. Journal of Mathematical Study, Vol. 53, No. 4, pp. 370-401,(2020).

[4] H. Chan and A. DelaTorre: "From fractional Lane-Emden-Serrin equation – existence, multiplicity and local behaviors via classical ODE – to fractional Yamabe metrics with singularity of "maximal" dimension." Preprint. Available at arXiv:2109.05647

[5] S. Cruz-Blázquez, A. DelaTorre, D. Ruiz. Qualitative properties of singular solutions of the fractional Yamabe problem. In preparation.

[6] A. DelaTorre and M.d.M González: Isolated singularities for a semilinear equation for the fractional Laplacian arising in conformal geometry. Revista Matemática Iberoamericana 34 (2018), no. 4.

[7] A. DelaTorre, M. del Pino, M.d.M González and J. Wei: "Delaunay-type singular solutions for the fractional Yamabe problem." Mathematische Annalen (2017),1-2

Boundary strong unique continuation property for fractional elliptic equations

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I will present a recent result contained in [2] of strong unique continuation property at boundary points for solutions to the following fractional elliptic equation

$$(-\Delta)^s u = hu$$

in a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, $s \in (0,1)$, under some outer homogeneous Dirichlet boundary condition.

The idea is to consider the Caffarelli-Silvestre extension (see [1]), thus providing an equivalent formulation of the fractional equation as a local problem in one dimension more. Then, after constructing a procedure of approximation of the domain, in the local context the classical approach developed by Garofalo and Lin in [3] allows to derive unique continuation from doubling conditions as a consequence of the boundedness of a suitable Almgren-type frequency function.

Combining the aforementioned analysis with blow-up arguments, a strong unique continuation can be achieved in the nonlocal setting as well.

References

[1] Caffarelli L., and Silvestre L. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations 32*, 7-9 (2007), 1245-1260.

[2] De Luca A., Felli V., and Vita S. Strong unique continuation and local asymptotics at the boundary for fractional elliptic equations. *Advances in Mathematics 400* (2022), 108279.

[3] Garofalo N., and Lin F.-H. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J. 35*, 2 (1986), 245-268.

Some results on the 3D Stokes eigenvalue problem under Navier boundary conditions

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We study the Stokes eigenvalue problem under Navier boundary conditions in $C^{1,1}$-domains $\Omega \subset \mathbb{R}^3$. Differently from the Dirichlet boundary conditions, zero may be the least eigenvalue. We fully characterize the domains where this happens and we show that the ball is the unique domain where the zero eigenvalue is not simple, it has multiplicity three. We apply these results to show the validity/failure of a suitable Poincaré-type inequality. The proofs are obtained by combining analytic and geometric arguments.

This is a joint work with Filippo Gazzola, Politecnico di Milano.

Minimal clusters in the plane with double densities

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The aim of this seminar is to present some results about the isoperimetric problem for clusters in the plane with double density. This amounts to finding the best configuration of m regions in the plane enclosing given volumes, in order to minimize their total perimeter, in the case where volume and perimeter are weighted by suitable densities.

We focus on the so-called "Steiner" property, ensuring that boundaries of minimal clusters are made of regular curves meeting in triple points. In the standard Euclidean case the directions at triple points are at 120 degrees. We show that the Steiner property can be generalized to a wide class of densities under natural assumptions. If the perimeter density is isotropic, *i.e.*, it does not depend on the normal, we show that triple points enjoy the usual 120 degrees Steiner property. For anisotropic densities the situation is more delicate and we discuss what possible directions occur in minimizers. Examples will be also discussed.

This is a joint collaboration with A. Pratelli and G. Stefani, see [1, 2].

References

[1] V. Franceschi, A. Pratelli, and G. Stefani, "*On the steiner property for planar minimizing clusters. the isotropic case.*," (arXiv 2106.08103).

[2] V. Franceschi, A. Pratelli, and G. Stefani, "*On the steiner property for planar minimizing clusters. the anisotropic case.*," (arXiv 2106.08099)

Consistency of the flat flow solution to the volume preserving mean curvature flow

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We consider the flat flow solution to the volume preserving mean curvature flow starting from $C^{1,1}$-regular set and show that it coincides with the classical solution as long as the latter exists. The proof is based on sharp regularity estimates for the minimizing movements scheme, that are stable with respect to the time discretization.

Long Time Behaviour of the Discrete Volume Preserving Mean Curvature Flow in the Flat Torus

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We show that the discrete approximate volume preserving mean curvature flow in the flat torus starting near a strictly stable critical set E of the perimeter converges in the long time to a translate of E exponentially fast. As an intermediate result we establish a new quantitative estimate of Alexandrov type for periodic strictly stable constant mean curvature hypersurfaces. Finally, in the two dimensional case a complete characterization of the long time behaviour of the discrete flow with arbitrary initial sets of finite perimeter is provided. This work has been done in collaboration with D. De Gennaro [1].

References

[1] D. De Gennaro and A. Kubin, *Long Time Behaviour of the Discrete Volume Preserving Mean Curvature Flow in the Flat Torus*, preprint 2022.

A spectral shape optimization problem with a nonlocal competing term

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We study the minimization under a measure constraint of a functional made as the sum of a cohesive spectral term and a repulsive Riesz-type interaction functional, namely

$$\min\Big\{\lambda_1(\Omega)+\varepsilon\int_\Omega\int_\Omega\frac{1}{|x-y|^{N-\alpha}}\,dx\,dy:\Omega\subset R^N,|\Omega|=1\Big\},$$

where $\alpha\in(1,N)$, $\varepsilon>0$ and λ_1 denotes the first eigenvalue of the Dirichlet Laplacian.

We show that there is a threshold $\varepsilon_1>0$ such that for all $\varepsilon\leq\varepsilon_1$ existence of minimizers occurs. Moreover we prove, by means of an expansion analysis, that the ball is a rigid minimizer. We also show that there is another threshold $\varepsilon_2>\varepsilon_1$ such that if $\varepsilon\geq\varepsilon_2$, then minimizers do not exist (at least in a suitable class of admissible sets).

The techniques and tools needed in the proofs are very broad. We employ spectral quantitative inequalities, the regularity of free boundaries, spectral surgery arguments and shape variations.

Moreover, we will present some open issues for similar classes of problems (motivated mostly by quantum mechanics models), which can be treated in this framework.

This is a joint project with Berardo Ruffini (Bologna).

The functional analytic approach for domain perturbation problems in spectral theory

Paolo Musolino

The functional analytic approach (FAA) is a method whose goal is to represent the solutions of perturbed boundary value problems in terms of real analytic maps and known functions of the perturbation parameters. Typical examples are regular domain perturbation problems, where a reference domain Ω is perturbed through a suitable diffeomorphism ϕ (and one considers a problem in the perturbed set $\phi(\Omega)$), or singular domain perturbations, where, for example, one perturbs Ω by making a small hole of size ϵ and then studies a certain boundary value problem in the perforated set Ω_ϵ. By exploiting the FAA, one would try to represent the solution of the boundary value problem in $\phi(\Omega)$ as a real analytic map of the diffeomorphism ϕ (which we think as a point in a certain Banach space) or of a problem in Ω_ϵ as a real analytic map of the size ϵ of the small perforation, for ϵ close to 0.

In this talk, we present some recent applications of the FAA to the study of the behavior of eigenvalues, both in the case of regular perturbations (see [2, 3]) and of singular perturbations (see [1]).

Based on joint works with L. Abatangelo, V. Bonnaillie-Noël, C. Léna, with P.D. Lamberti, P. Luzzini, and with M. Lanza de Cristoforis, J. Taskinen.

References

[1] L. Abatangelo, V. Bonnaillie-Noël, C. Léna, P. Musolino, Asymptotic behavior of u-capacities and singular perturbations for the Dirichlet-Laplacian. *ESAIM Control Optim. Calc. Var.* 27 (2021), suppl., Paper No. S25, 43 pp.

[2] P.D. Lamberti, P. Luzzini, P. Musolino, Shape perturbation of Grushin eigenvalues. *J. Geom. Anal.* 31 (2021), no. 11, 10679–10717.

[3] M. Lanza de Cristoforis, P. Musolino, J. Taskinen, A real analyticity result for symmetric functions of the eigenvalues of a quasiperiodic spectral problem for the Dirichlet Laplacian. *J. Operator Theory* 86 (2021), no. 2, 439–467.

Resolution of singularities of the network flow

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The curve shortening flow is an evolution equation in which a curve moves with normal velocity equal to its curvature (at any point and time) and can be interpreted as the gradient flow of the length. We consider the same flow for networks (finite unions of sufficiently smooth curves whose end points meet at junctions). Because of the variational nature of the problem, one expects that for almost all the times the evolving network will possess only triple junctions where the unit tangent vectors forms angles of 120 degrees (regular junctions). However, even if the initial network has only regular junctions, this property is not preserved by the flow and junctions of four or more curves may appear during the evolution. The aim of this talk is first to describe the process of singularity formation and then to explain the resolution of such singularities and how to continue the flow in a classical PDE framework.

This is a research in collaboration with Jorge Lira (Federal University of Ceará), Rafe Mazzeo (Stanford University) and Mariel Saez (Pontificia Universidad Católica de Chile).

Optimal Leray-Trudinger inequality

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Let $\Omega \subset \mathbf{R}^n$, $n \in \mathbf{N} \setminus \{1\}$, be a bounded domain that contains the origin and set $R_\Omega := \sup_{x\in\Omega} |x|$. A critical case of Hardy's inequality asserts that

$$I_n[u;\Omega] := \int_\Omega |\nabla u(x)|^n \, \mathrm{d}x - \Big(\frac{n-1}{n}\Big)^n \int_\Omega \frac{|u(x)|^n}{|x|^n} X^n\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x \geq 0,$$

for all $u \in W_0^{1,n}(\Omega)$. Here, $X(t) := (1-\log t)^{-1}$, $t \in [0,1]$. This is a scale invariant inequality where the power n on X cannot be decreased and the constant $((n-1)/n)^n$ cannot be increased. On the other hand, a well known theorem of Trudinger asserts that with $\mathcal{D} := \big\{u \in W_0^{1,n}(\Omega) \bigm| \||\nabla u|\|_{L^n(\Omega)} \leq 1\big\}$,

$$\sup_{u\in\mathcal{D}} \frac{\|u\|_{L^q(\Omega)}}{q^{1-1/n}} < \infty \text{ for all } q > n \text{ and } 1 - 1/n \text{ cannot be increased.}$$

In our recent work with Giuseppina Di Blasio and Giovanni Pisante we found out what is the optimal analogue of Trudinger's theorem if one replaces $\mathcal{D}$ by $\mathcal{H} := \big\{u \in W_0^{1,n}(\Omega) \bigm| I_n[u;\Omega] \leq 1\big\}$. My talk is about the origin and history of this problem and also a discussion on the method of proof.

References

[1] G. Di Blasio, G. Pisante, G. Psaradakis, *A sharp Leray-Trudinger inequality*, *preprint* (2022).

[2] S. Filippas, A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal. 192 (2002) 186-233.

[3] K. T. Gkikas, G. Psaradakis, *Optimal non-homogeneous improvements for the series expansion of Hardy's inequality*, Commun. Contemp. Math. *online ready* (2021) 2150031.

[4] A. Mallick, C. Tintarev, *An improved Leray-Trudinger inequality*, Commun. Contemp. Math. 20 (2017) 1750034.

[5] G. Psaradakis, *An optimal Hardy-Morrey inequality*, Calc. Var. Partial Differential Equations 45 (2012) 421-441.

[6] G. Psaradakis, D. Spector, *A Leray-Trudinger inequality*, J. Funct. Anal. 269 (2015) 215-228.

Symmetry results for the critical $p-$Laplace equation

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We consider the critical p-Laplace equation:

$$\Delta_p u + u^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n \,, \tag{1}$$

with $n \geq 2$ and $1 < p < n$. Equation (1) has been largely studied in the PDE's and geometric analysis' communities, since extremals of Sobolev inequality solve (1) and, for $p = 2$, the equation is related to the Yamabe's problem. In particular it has been recently shown, exploiting the moving planes method, that positive solutions to (1) with finite energy, i.e. such that $u \in L^{p^*}(\mathbb{R}^n)$ and $\nabla u \in L^p(\mathbb{R}^n)$, can be completely classified. In this talk, we present some recent classification results for positive solutions to (1) with (possibly) infinite energy satisfying suitable conditions at infinity.

This is based on a recent joint work with G. Catino and D. Monticelli.

Principal spectral curves for Lane-Emden fully nonlinear type systems and applications

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I will present some recent results, obtained in collaboration with Ederson Moreira dos Santos, Gabrielle Nornberg and Hugo Tavares, about spectral properties of fully nonlinear Lane-Emden type systems with possibly unbounded coefficients and weights. In particular, we prove the existence of two principal spectral curves on the plane. We also construct a possible third spectral curve related to a second eigenvalue and an anti-maximum principle, which are novelties even for standard Lane-Emden systems involving the Laplacian operator. Most of the results are new even in the scalar case, in particular for Isaac's operators with unbounded coefficients.