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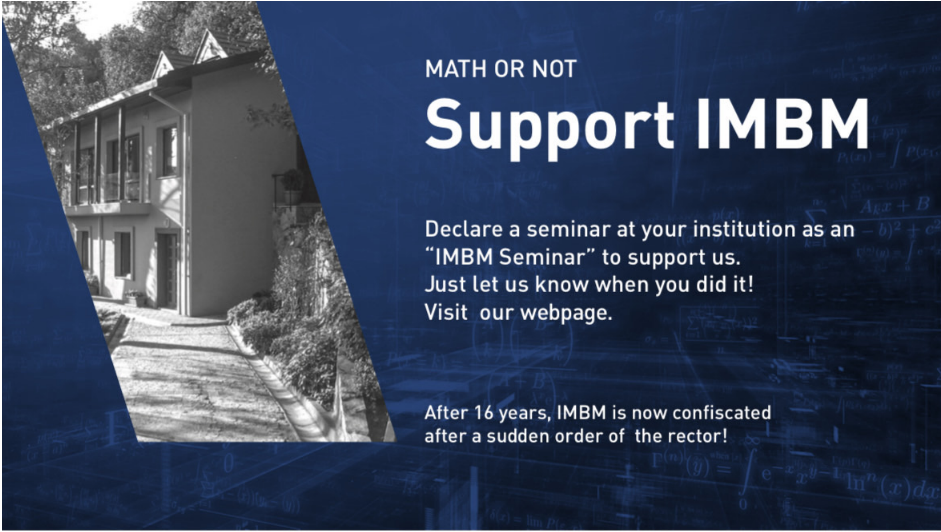

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Model Companions and Group Actions on Fields

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PART I
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Model Completeness

Definition

Let T be an $\mathcal{L}$-theory and let $\mathcal{A}$ and $\mathcal{B}$ be models of T.
T is called **model complete** if whenever $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{A} \preccurlyeq \mathcal{B}$.

Definition

Let T be an $\mathcal{L}$-theory. A model $\mathcal{A}$ of T is called **existentially closed** model of T if for any extension $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B} \models T$ and for any quantifier free $\mathcal{L}_A$-formula $\phi(\bar{v})$, we have the following:

$$\mathcal{A} \models \exists \bar{v}\, \phi(\bar{v}) \quad \text{if and only if} \quad \mathcal{B} \models \exists \bar{v}\, \phi(\bar{v}.)$$

Model Completeness

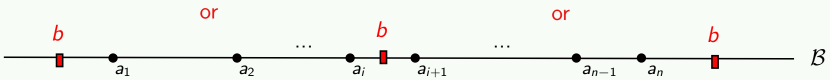
Equivalent Conditions of Model Completeness

A theory T is **model complete** if one of the equivalent conditions below is satisfied.

- Every model of T is **existentially closed**.
- Every embedding between models of T are elementary embedding.
- For any model $\mathcal{A}$ of T, $T \cup Diag(\mathcal{A})$ is complete.
- For every formula $\phi(\bar{v})$, there is a universal formula $\psi(\bar{v})$ such that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

Example 1

The theory of dense linear orders without endpoints (DLO) is **model complete**. Let $\mathcal{A} \subseteq \mathcal{B}$, an existential $\mathcal{L}_A$-sentence $\phi(\bar{a})$ can only describe position of an element relative to a_i's.



Example 2

The theory of dense linear orders **with** endpoints is **not model complete**.

Consider two models $\mathcal{A} = ([0,1], <)$ and $\mathcal{B} = ([0,2], <)$.

$\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{B} \models \exists v(1 < v)$; but, $\mathcal{A} \not\models \exists v(1 < v)$.

Example 3

The theory of algebraically closed fields is **model complete**, as every theory with quantifier elimination.

Model Companion

Definition

Let T be an $\mathcal{L}$-theory. An $\mathcal{L}$-theory T^* is called **model companion** of T if the following three conditions are satisfied:

- T^* is model complete.
- Every model of T can be embedded into a model of T^*.
- Every model of T^* can be embedded into a model of T.

- A theory is called **companionable** if it has a model companion
- If a theory T is companionable, then a model companion T^* of T is unique up to equivalence of theories.

Inductive Theories

Definition

An $\mathcal{L}$-theory T is called **inductive** if for any chain $(\mathcal{A}_i : i \in I)$ of models of T (i.e. $(I, <)$is a linearly ordered set, $\mathcal{A}_i \subseteq \mathcal{A}_j$ for any $i < j$), we have $\bigcup_{i \in I} \mathcal{A}_i \models T$.

Example

(i) *The theories of fields, groups, rings are inductive.*

(ii) *The theory of dense linear orders with endpoints is not inductive. Consider the chain of models* $(\mathcal{A}_i : i \geq 1)$ *where* $\mathcal{A}_i = ([-i, i], <)$.

Theorem

Let T be an inductive theory.

- *T is inductive if and only if T is a $\forall\exists$-theory.*
- *Every model of T can be extended to an existentially closed model of T.*
- *If the model companion T^* of T exists, then models of T^* are* **exactly the existentially closed models** *of T.*
- *T is companionable if and only if the class of existentially closed models is elementary.*

Positive Examples

1. The model companion of the theory of sets is the theory of infinite sets.

2. The model companion of the theory of equivalence relations is the theory of equivalence relations with infinitely many infinite classes.

3. The model companion of the theory of linear orders is the theory of dense linear orders without endpoints.

4. The model companion of the theory of graphs is the theory of random graph.

5. The model companion of the theory of fields is the theory of algebraically closed fields.

6. The model companion of the theory of rings without nilpotents exists. (Lipshitz L., Saracino D).

When Model Companions Do Not Exist

- The theory of groups do not have model companion (Ekloff-Sabbagh).
- The theory of commutative rings do not have model companion (Cherlin).
- The theory of cycle free graphs do not have model companion (Naito).
- $(\mathbb{Z} \times \mathbb{Z})$-TCF does not exists, i.e. existentialy closed $\mathbb{Z} \times \mathbb{Z}$ fields do not have model companion (Hrushovski).
- The theory of dense linear orders with an automorphism has no model companion (Kikyo-Shelah).
- $(\mathbb{Z} \rtimes \mathbb{Z})$-TCF does not exists (B.-Kowalski).

Obstacle Argument

There is a common theme in each of the above proofs.

Definition

Let T be an $\mathcal{L}$-theory and $\phi(\bar{v})$ be an $\mathcal{L}$-formula. An $\mathcal{L}$-formula $\psi(\bar{v})$ is called a ϕ-**obstacle** if $T \cup \{\phi(\bar{v})\} \cup \{\psi(\bar{v})\}$ is "inconsistent"; that is, if there is no model $\mathcal{A}$ of T with a tuple $\bar{a} \in A^n$ such that $\mathcal{A} \models \phi(\bar{a}) \wedge \psi(\bar{a})$.

Example

Let $\mathcal{R} = (\mathbb{R}, +, \cdot, -, 0, 1, <)$ be the ordered field of real numbers and let $T = RCF$ be the theory of real closed fields.

$$\phi(v) : \exists z \, (v = z^2) \qquad \phi\text{-}\textit{obstacle}\ \psi(v) : (v < 0)$$

Obstacle Argument (Takeuchi, Tanaka and Tsuboi, 2015)

Let T be an inductive $\mathcal{L}$-theory. If there is an existential $\mathcal{L}$-formula $\phi(\bar{v})$ and a set of $\mathcal{L}$-formulas $\Sigma(\bar{v})$ such that:

(i) For any existentially closed model $\mathcal{A}$ of T and for all $\bar{a} \in A^n$, we have $\mathcal{A} \models \Sigma(\bar{a}) \to \mathcal{A} \models \phi(\bar{a})$.

(ii) For any finite subset $\Sigma_0(\bar{v})$ of $\Sigma(\bar{v})$, there is an existentially closed model $\mathcal{B}$ of T and a ϕ-obstacle $\psi(\bar{v})$ (depending on $\Sigma_0(\bar{v})$) such that $\mathcal{B} \models \Sigma_0(\bar{b})$ and $\mathcal{B} \models \psi(\bar{b})$ for some $\bar{b} \in B^n$.

Then, T has no model companion.

Negative Examples

Theorem (Eklof and Sabbagh, 1971)

The theory of groups has no model companion.

HNN-extensions

Two elements of a group G have the same order if and only if they are conjugate in some group extension H of G.

- $\Sigma(v_1, v_2)$: $\{v_1^n \neq e, v_2^n \neq e : n > 0\}$ (infinite order)
- $\phi(v_1, v_2)$: $\exists w\,(v_1 \cdot w = w \cdot v_2)$ (conjugate)
- $\Sigma_0(v_1, v_2)$: $\{v_1^n \neq e, v_2^n \neq e : 0 < n < N\}$
- $\psi(v_1, v_2)$: $v_1^N = e \wedge v_2^N \neq e$ is a ϕ-obstacle for $\Sigma_0(v_1, v_2)$

Negative Examples

Theorem (Cherlin, 1973)

The theory of commutative rings has no model companion.

Lemma

Let R be a commutative ring and let $r \in R$. TFAE:
(i) r is not nilpotent; that is, $r^n \neq 0$ for any $n \in \mathbb{N}$.
(ii) There is a commutative ring extension R' of R and a nonzero *idempotent element* a (*i.e.*, $a^2 = a$) of R' such that r divides a in R'.

- $\Sigma(v) : \{v^n \neq 0 : n > 0\}$
- $\phi(v) : \exists w_1 \exists w_2 \, [(w_1^2 = w_1) \wedge (w_1 \neq 0) \wedge (v \cdot w_2 = w_1)]$
- $\Sigma_0(v) : \{v^i \neq 0 : 0 < i < N\}$
- $\psi(v) : v^N = 0$ is a ϕ-obstacle

Fields with two commuting automorphisms

Theorem (Hrushovski)

The theory of fields with two commuting automorphisms has no model companion.

- $\Sigma(v) = \{(\sigma(v) = \tau(v)) \wedge (\sigma^n(v) + \sigma^{n-1}(v) + \ldots + \sigma(v) + v \neq 0) : n \in \mathbb{N}\}$
- $\phi(v) : \exists z(z^3 = 1 \wedge z \neq 1 \wedge \sigma(z) = \tau(z) = z^2) \rightarrow$
  $\exists w_1 \exists w_2 \, [(\sigma(w_1) = \tau(w_1) = w_1 + v) \wedge (w_2^3 = w_1) \wedge (\tau(w_2) = z\sigma(w_2))]$
- $\Sigma_0(v) = \{\sigma(v) = \tau(v) \wedge \sigma^i(v) + \sigma^{i-1}(v) + \ldots + \sigma(v) + v \neq 0 : i < N\}$
- $\psi(v) : (v + \sigma(v) + \sigma^2(v) + \ldots + \sigma^{m-1}(v) = 0) \wedge (\sigma(\zeta) = \tau(\zeta) = \zeta^2)$
  is a ϕ-obstacle if $m > N$ is odd.

PART II
with Piotr Kowalski

G-fields as first order structures

- Given a group G, a G-**field** is a field K together with an action of G on K by automorphisms.
- A G-field can be considered as a first-order structure $(K, +, \cdot, (\sigma_g)_{g\in G})$, where σ_g is the automorphism determined by the action of the element $g \in G$.
- It is also enough to name the automorphisms σ_g for g in some chosen set of generators.
- We define G-**field extensions**, G-**rings**, etc. in a natural way.

Existentially closed G-fields

- The theory of G-fields consists of field axioms, plus the axioms stating that the σ_g's are field automorphisms, and $g \mapsto \sigma_g$ is a group action.
- Note that this theory is inductive, i.e. $\forall\exists$-axiomatized.
- Also note that all the σ_g's may act as the identity automorphism, even though the group G is not trivial.
- Nevertheless, if we consider an **existentially closed G-field K**, then the action of G on K is faithful.
- We will focus our attention on existentially closed G-fields.

Properties of existentially closed G-fields

- Any G-field has an e.c. G-field extension. (A general property of inductive theories.)
- For $G = \{1\}$, e.c. G-fields coincide with algebraically closed fields.
- For $G = \mathbb{Z}$, e.c. G-fields coincide with *transformally* (or *difference*) *closed fields*.
- Existentially closed G-fields are not necessarily algebraically closed.
- The complex field $\mathbb{C}$ with the complex conjugation automorphism is not an e.c. C_2-field. (C_n is the cyclic group of n elements.)

Properties of existentially closed G-fields (Sjögren)

Field theoretic properties

Let K be an e.c. G-field and let $F = K^G$ be the fixed field of G.

- G acts faithfully on the field K.
- Both K and F are perfect.
- Both K and F are pseudo algebraically closed (PAC), hence their absolute Galois groups are projective pro-finite groups.

Galois theoretic properties

If we also assume that G is finitely generated then:

- $\mathrm{Gal}(F^{\mathrm{alg}} \cap K/F)$ is the profinite completion $\hat{G}$ of G.
- The absolute Galois group of F is the universal Frattini cover $\widetilde{\hat{G}}$ of the profinite completion $\hat{G}$ of G.

Profinite groups as Galois groups: Some definitions

- A **profinite group is projective**, if for every profinite A, B with $\alpha : A \to B$ onto, and $\beta : G \to B$, there exists $\gamma : G \to A$ such that the diagram
$$\begin{tikzcd} & G \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \\ A \arrow[r, two heads, "\alpha"] & B \end{tikzcd}$$
commutes.
- A profinite group is called **small** if it has only finitely many open subgroups of index n for each positive integer n.
- **Frattini Cover**: An epimorphism $\varphi : H \to G$ of profinite groups is called a Frattini cover if for all closed $H_0 < H$, if $\varphi(H_0) = G$ then $H_0 = H$.
- **Universal Frattini Cover** is a projective Frattini cover, and it is unique up to isomorphism.

Galois Axioms for finite G

Theorem (Sjögren, and independently by Hoffmann-Kowalski)

Let G be a finite group. A G-field (K, G) is existentially closed iff

- *the fixed field $F = K^G$ is PAC,*
- *the action of G is faithful,*
- *the restriction $\mathrm{Gal}(F) \to \mathrm{Gal}(K/F) = G$ is a Frattini cover of G.*

- The above theorem provides an axiomatization for the theory G-TCF when G is finite. We will call this kind of conditions: **Galois axioms**.
- We may consider Galois axioms for an arbitrary group G, and existentially closed G-fields satisfy them. However they are usually not enough to axiomatize G-TCF (which may not even exist!).
- Finiteness of G is essential!

Torsion Abelian Groups

Here is the final version of our theorem about torsion abelian group actions.

Theorem (B. Kowalski)

Let A be a torsion Abelian group. Then $A-\mathrm{TCF}$ exists if and only if for each prime p, the p primary part of A is either of Prüfer rank one or finite.

The following two steps are conceptually crucial for proving the general case:

1. the theory $C_{p^\infty}-\mathrm{TCF}$ exists;
2. the theory $C_{p^\infty}^2-\mathrm{TCF}$ does not exist.

Results we have been announcing (Final Version)

We presented the following results in several talks (with varying degrees of probability of correctness) for existence or non-existence of G-TCF. Here is the final version.

- If $G = C_{p^\infty}$ is the Prüfer p-group, then G-TCF exists.
- If $G = C_{p_1^\infty} \oplus C_{p_2^\infty} \oplus \ldots \oplus C_{p_n^\infty}$ where p_i's are distinct primes, then G-TCF exists.
- If $G = C_p^{(\omega)}$ (direct sum of infinitely many C_p's), then G-TCF **does not** exist.
- If $G = C_2 \oplus C_3 \oplus C_5 \oplus \ldots =: C_{\mathbb{P}}$ direct sum of finite cyclic groups of prime order, then G-TCF exists.

$G = C_{p^\infty}$

Theorem

Let C_{p^∞} be the Prüfer p-group, then C_{p^∞}-TCF exists.

- Let C_{p^∞} be the direct limit of the groups $C_{p^n} = \langle \sigma_n \rangle$ with the embeddings given by $\sigma_n \mapsto \sigma_{n+1}^p$.
- It is enough to show that: if (K, σ_{n+1}) is a model of $C_{p^{n+1}}$-TCF, then (K, σ_{n+1}^p) is a model of C_{p^n}-TCF.
- We only need to check: $F = \mathrm{Fix}(\sigma_{n+1}^p)$ is PAC, and $\mathrm{Gal}(F) \to \mathrm{Gal}(K/F) = C_{p^n}$ is a Frattini cover of C_{p^n}.
- F is a finite extension of the PAC field $\mathrm{Fix}(\sigma_{n+1})$, hence it is PAC.
- $\mathrm{Gal}(F) = p\mathbb{Z}_p \cong \mathbb{Z}_p$ is the universal Frattini cover of C_{p^n}.
- Hence C_{p^n}-TCF can be understood as a subtheory of $C_{p^{n+1}}$-TCF, and our theory C_{p^∞}-TCF is the increasing union of the theories C_{p^n}-TCF.

$G = C_{p^\infty} \oplus C_{p^\infty}$

- Note that the above construction can be extended to finite products of Prüfer groups for non repeating primes. I.e. for distinct primes $p_1, p_2, \ldots, p_n$, the theory $C_{p_1^\infty} \oplus C_{p_2^\infty} \oplus \ldots \oplus C_{p_n^\infty}$-TCF exists.
- But the same method does not provide a model companion in the case $G = C_{p^\infty} \oplus C_{p^\infty}$ (below). Actually this model companion does not exist.
- Given a model (K, σ, τ) of $C_{p^2} \oplus C_{p^2}$, (K, σ^p, τ^p) is a $C_p \oplus C_p$-field.
- Let us set $F := \mathrm{Fix}(\sigma, \tau)$, $F' := \mathrm{Fix}(\sigma^p, \tau^p)$. We have

$$\mathrm{Gal}(F) = \widehat{F_2}(p), \quad [\mathrm{Gal}(F) : \mathrm{Gal}(F')] = p^2.$$

  By profinite Nielsen-Schreier formula, $\mathrm{Gal}(F') = \widehat{F_m}(p)$ where $m = 1 + p^2$. Hence $\mathrm{Gal}(F')$ is not a Frattini cover of $C_p \oplus C_p$ and $C_p \oplus C_p$-TCF does not embed into $C_{p^2} \oplus C_{p^2}$-TCF.