Locally compact models for approximate rings

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Approximate groups

Definition

A symmetric subset X of a group is an *approximate subgroup* if $X^2 \subseteq FX$ for some finite $F \subseteq \langle X \rangle$. It is *definable* in M if $X, X^2, X^3, \ldots$ are all definable in M and the restrictions $\cdot|_{X^n \times X^n} \colon X^n \times X^n \to X^{2n}$ are all definable in M.

Definition

A *locally compact model* of an approximate subgroup X is a group homomorphism $f \colon \langle X \rangle \to H$ to some locally compact group H s.t.:

1. $f[X]$ is relatively compact in H,
2. $f^{-1}[U] \subseteq X^m$ for some $m < \omega$ and $U \subseteq H$ an open neighborhood of e.

In the definable context, we additionally require *definability* of f:

3. For any $C \subseteq U \subseteq H$ where C is compact and U is open, there exists a definable Y such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$.

Approximate groups — Hrushovski's theorem

Theorem (Hrushovski)

A pseudofinite approximate subgroup has a locally compact model with $m = 4$.

Using Yamabe's theorem

Corollary (Hrushovski)

For a pseudofinite approximate subgroup X there is a commensurable approximate subgroup $Y \subseteq X^4$ which has a Lie model.

This paved the way for Breuillard, Green, and Tao to give a full classification of all finite approximate subgroups

Approximate groups — G^{00}.

Let X be a definable approximate subgroup, and $\bar{X}$ its interpretation in a monster model. Let $G := \langle X \rangle$ and $\bar{G} := \langle \bar{X} \rangle$.

Fact

TFAE

1. A definable locally compact model of X exists.
2. There exists an M-type-definable subgroup of $\bar{G}$ of bounded index.
3. There exists the smallest M-type-definable subgroup of $\bar{G}$ of bounded index, which is denoted by $\bar{G}^{00}_M$.

Remark

If $\bar{G}^{00}_M$ exists, then $\bar{G}^{00}_M \subseteq \bar{X}^m$ for some $m < \omega$. The last inclusion is equivalent to the existence of definable, symmetric, generic subsets D_n, $n < \omega$, of X^m with $D_{n+1}D_{n+1} \subseteq D_n$ for all n.

Approximate groups — G^{00}.

Proposition

If $\bar{G}^{00}_M$ exists, then the quotient map $G \to \bar{G}/\bar{G}^{00}_M$ is the universal definable locally compact model of X.

Here, $F \subseteq \bar{G}/\bar{G}^{00}_M$ is closed if $\pi^{-1}[F] \cap \bar{X}^n$ is type-definable for every $n < \omega$, where $\pi \colon \bar{G} \to \bar{G}/\bar{G}^{00}_M$ is the quotient map.

Example (Hrushovski, Krupiński, Pillay)

Let $f \colon \mathbb{F}_{a,b} \to \mathbb{Z}$ be the quasi-homomorphism given by

$$f(a^{n_1}b^{m_1}\dots a^{n_k}b^{m_k}) := \sum_{i=1}^{k} \operatorname{sgn}(n_i) + \operatorname{sgn}(m_i)$$

Then $X := \operatorname{graph}(f)$ is an approximate subgroup definable in $M := (\mathbb{F}_{a,b}, (\mathbb{Z}, +), f)$ or in any expansion of it, for which $\bar{G}^{00}_M$ does not exist, so there is NO locally compact model of X.

Approximate rings — definition

Definition

An additively symmetric subset X of a ring is an *approximate subring* if $XX \cup (X + X) \subseteq F + X$ for some finite subset F of the ring $\langle X \rangle$ generated by X.

Example

1. $X := [-1, 1]$ is an approximate subring of $\mathbb{R}$.
2. $X := \{\sum_{i=-1}^{\infty} a_i t^i : a_i \in \mathbb{F}_p\}$ is an approximate subring of the field of formal Laurent series $\mathbb{F}_p((t))$ over the finite field $\mathbb{F}_p$.

For X an approximate subring we recursively define: $X_0 := X$, $X_{n+1} := X_n X_n + (X_n + X_n)$.

Definition

An approximate subring X is *definable* in M if all X_n's are definable in M and the restrictions of $+$ and $\cdot$ to any X_n are definable in M.

Locally compact models

Definition

A *locally compact model* of an approximate subring X is a ring homomorphism $f \colon \langle X \rangle \to S$ to some locally compact ring S s.t.:

1. $f[X]$ is relatively compact in S,
2. $f^{-1}[U] \subseteq X_m$ for some $m < \omega$ and $U \subseteq S$ an open neighborhood of e.

In the definable context, we additionally require *definability* of f:

1. For any $C \subseteq U \subseteq S$ where C is compact and U is open, there exists a definable Y such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$.

Locally compact models — cont.

Let X be a definable approximate subring, and $\bar{X}$ its interpretation in a monster model. Let $R := \langle X \rangle$ and $\bar{R} := \langle \bar{X} \rangle$.

Proposition

TFAE

1. A definable locally compact model of X exists.
2. There exists an M-type-definable two-sided ideal of $\bar{R} := \langle \bar{X} \rangle$ of bounded index.
3. There exists the smallest M-type-definable two-sided ideal of $\bar{R}$ of bounded index, which is denoted by $\bar{R}^{00}_M$.

Proposition

If $\bar{R}^{00}_M$ exists, then $\bar{R}^{00}_M \subseteq \bar{X}_m$ for some $m < \omega$. The last inclusion is equivalent to the existence of definable, symmetric, additively generic subsets D_n, $n < \omega$, of X_m with $D_{n+1}D_{n+1} + (D_{n+1} + D_{n+1}) \subseteq D_n$ for all n.

Locally compact models — cont.

Proposition

If $\bar{R}_M^{00}$ exists, then the quotient map $R \to \bar{R}/\bar{R}_M^{00}$ is the universal definable locally compact model of X.

Here, $F \subseteq \bar{R}/\bar{R}_M^{00}$ is closed if $\pi^{-1}[F] \cap \bar{X}_n$ is type-definable for every $n < \omega$, where $\pi \colon \bar{R} \to \bar{R}/\bar{R}_M^{00}$ is the quotient map.

Main goal

$\bar{R}_M^{00}$ always exists, and so $R \to \bar{R}/\bar{R}_M^{00}$ is the universal definable locally compact model of X.

Model-theoretic components of rings

Model-theoretic components of definable rings were defined and studied in [GJK] and later in [KR]. The main application in [GJK] was a computation of definable (in particular classical) Bohr compactifications of some matrix groups. The present work yields another application (but this time of model-theoretic components of rings generated by definable approximate subrings), namely to show the existence of locally compact models of arbitrary approximate subrings.

[GJK] J. Gismatullin, G. Jagiella, K. Krupiński, *Bohr compactifications of groups and rings*, J. Symb. Log., accepted.

[KR] K. Krupiński, T. Rzepecki, *Generating ideals by additive subgroups of rings*, Ann. Pure App. Logic (173), 103119, 2022.

Let R be a 0-definable group [resp. ring], $\mathfrak{C}$ a monster model, $\bar{R} = R(\mathfrak{C})$, and $A \subseteq \mathfrak{C}$ be a small set of parameters.

- $\bar{R}^0_A$ is the intersection of all A-definable, finite index subgroups [ideals] of $\bar{R}$.
- $\bar{R}^{00}_A$ is the smallest A-type-definable, bounded index subgroup [ideal] of $\bar{R}$.
- $\bar{R}^{000}_A$ is the smallest A-invariant, bounded index subgroup [ideal] of $\bar{R}$.

Proposition ([GJK])

The above components of the ring $\bar{R}$ exist and do not depend on the choice of the version (left, right, or two-sided) of the ideals. Moreover, instead of "ideal" we can equivalently write "subring" in the above definitions.

Generating ideals in finitely many steps

Theorem 1 ([KR])

Let R be an arbitrary ring 0-definable in a structure M and $A \subseteq M$. Then for every A-definable finite index subgroup H of $(R,+)$, the set $H + R \cdot H$ contains an A-definable, two-sided ideal of R of finite index.

Theorem 2 ([KR])

Let R be a 0-definable ring and $A \subseteq \mathfrak{C}$ a small set of parameters.

1. $(\bar{R},+)^0_A + \bar{R} \cdot (\bar{R},+)^0_A = \bar{R}^0_A$.
2. If R is unital, then $(\bar{R},+)^{00}_A + \bar{R} \cdot (\bar{R},+)^{00}_A + \bar{R} \cdot (\bar{R},+)^{00}_A = \bar{R}^{000}_A = \bar{R}^{00}_A = \bar{R}^0_A$.
3. If R is of positive characteristic (not necessarily unital), then $(\bar{R},+)^{00}_A + \bar{R} \cdot (\bar{R},+)^{00}_A = \bar{R}^{000}_A = \bar{R}^{00}_A = \bar{R}^0_A$.

Examples

Example ([KR])

There is a subgroup H of $\mathbb{Z}[X]$ of index 4 such that $R \cdot H$ does not contain an ideal of finite index. So $1\frac{1}{2}$ steps in Theorems 1 and 2(1) is optimal (i.e. cannot be decreased). Also, $2\frac{1}{2}$ steps in Thm. 2(2) cannot be decreased to 1 step. We also constructed an example of characteristic 2, showing that $1\frac{1}{2}$ steps is optimal in Thm. 2(3).

Example ([KR])

Let $R := \mathbb{Z}_2^{\omega}$ be equipped with the full structure. There is a type-definable (in fact the intersection of a countable descending sequence of 0-definable subgroups) bounded index subgroup H of the additive group of $\bar{R}$ such that $\bar{R} \cdot H$ does not generate a group in finitely many steps.

Model-theoretic components of rings generated by definable approximate subrings

Let X be a 0-definable (in M) approximate subring, $R := \langle X \rangle$, $\mathfrak{C}$ a monster model, $\bar{X} := X(\mathfrak{C})$, $\bar{R} := \langle \bar{X} \rangle$, and A a small subset of $\mathfrak{C}$. We define $\bar{R}^{00}_A$ and $\bar{R}^{000}_A$ as for definable R.
By arguments from [GJK], we get:

Proposition

1. $\bar{R}^{000}_A$ exists and does not depend on the choice of the version (left, right, or two-sided) of the ideals. Moreover, instead of "ideal" we can equivalently write "subring" in the above definition.
2. The definition of $\bar{R}^{00}_A$ does not depend on the choice of the version (left, right, or two-sided) of the ideals. Moreover, instead of "ideal" we can equivalently write "subring" in the above definition. However, the existence of $\bar{R}^{00}_A$ is a nontrivial issue.

Main results

Main Theorem

$(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A = \bar{R}^{000}_A$. Moreover, if $R \subseteq \mathrm{dcl}(A)$ (e.g. $A = R$ or $A = M$), then $\bar{R}^{00}_A$ exists and equals $\bar{R}^{000}_A = (\bar{R},+)^{00}_A + \bar{X}(\bar{R},+)^{00}_A$.

Corollary

If $\bar{R}$ is definable, then $(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A = \bar{R}^{000}_A = \bar{R}^{00}_A$ for an arbitrary small $A \subseteq \mathfrak{C}$.

Main Corollary

X has a definable locally compact model. More precisely, the quotient map $h\colon R \to \bar{R}/\bar{R}^{00}_M$ is the universal definable locally compact model of X, and $U := \{a/\bar{R}^{00}_M : a + \bar{R}^{00}_M \subseteq 4\bar{X} + \bar{X}\cdot 4\bar{X}\}$ is an open neighborhood of $0/\bar{R}^{00}_M$ such that $h^{-1}[U] \subseteq 4X + X\cdot 4X$.

Ingredients of the proof

The following fact follows from results of Massicot and Wagner.

Fact

If Z is a definably amenable (e.g. abelian) 0-definable (in M) approximate subgroup, then $\langle \bar{Z} \rangle_A^{00}$ exists (where $\langle \bar{Z} \rangle$ is the group generated by $\bar{Z}$). Moreover, $\langle \bar{Z} \rangle_A^{00} \subseteq \bar{Z}^8$, and if $A = M$, then $\langle \bar{Z} \rangle_A^{00} \subseteq \bar{Z}^4$. In particular, $(\bar{R}, +)_A^{00}$ exists and is contained in $8\bar{X}$, and if $A = M$, then it is contained in $4\bar{X}$.

Definition

A definable, additively symmetric subset D of $\bar{R}$ is *thick* if for every sequence $(r_i)_{i<\lambda}$ of unbounded (equiv. uncountable) length which consists of elements of $\bar{R}$ there are $i < j < \lambda$ with $r_j - r_i \in D$.

Remark

$(\bar{R}, +)_A^{00}$ is the intersection of a downward directed family of A-definable thick subsets of $\bar{R}$.

Ingredients of the proof — cont.

Definition

Two subgroups H_1 and H_2 of an abelian group G are *coset-independent* if any coset of H_1 intersects any coset of H_2.

Put $H := (\bar{R}, +)^{00}_A$. Working with thick sets and coset-independence, we prove

Main technical lemma

Let G be the intersection of all sets of the form $\bar{R}K/H$, where K ranges over all bounded index subgroups of $(\bar{R}, +)$ which are type-definable over some sets of parameters of cardinality at most $2^{2^{|\mathcal{L}|+|A|}}$. Then G is a subgroup of $(\bar{R}/H, +)$.

Ingredients of the proof — cont.

The next fact follows from results of Massicot and Wagner as observed by Pillay and myself for definable groups; for $G := \langle \bar{Z} \rangle$, one needs to use a basic observation of Hrushovski, Pillay and myself that G_A^{000} is generated by the intersection of all A-definable thick subsets of G.

Fact

If Z is a definably amenable 0-definable approximate subgroup, then $\langle \bar{Z} \rangle_A^{00} = \langle \bar{Z} \rangle_A^{000}$. In particular, $(\bar{R}, +)_A^{00} = (\bar{R}, +)_A^{000}$.

Then the main theorem can be proved using the main technical lemma and the above fact. The idea is that from these results, we deduce that G from the lemma coincides with $\bar{R}H/H$, and so $H + \bar{R}H$ is an A-invariant left ideal of bounded index. Thus,

$$\bar{R}_A^{000} \subseteq H + \bar{R}H = (\bar{R}, +)_A^{000} + \bar{R}(\bar{R}, +)_A^{000} \subseteq \bar{R}_A^{000} + \bar{R}\bar{R}_A^{000} = \bar{R}_A^{000},$$

and hence $H + \bar{R}H = \bar{R}_A^{000}$ as required.

Ingredients of the proof — cont.

For the "moreover" part, it is enough to show that

$$(\bar{R},+)^{00}_A + \bar{R}(\bar{R},+)^{00}_A = (\bar{R},+)^{00}_A + \bar{X}(\bar{R},+)^{00}_A.$$

$(\supseteq)$ is obvious. To show $(\subseteq)$, we choose a countable $Y \subseteq R$ s.t. $Y + \bar{X} = \bar{R}$. It remains to show that

$$(\forall y \in Y)(y(\bar{R},+)^{00}_A \subseteq (\bar{R},+)^{00}_A).$$

The last inclusion follows easily from the fact that $y \in R \subseteq \operatorname{dcl}(A)$.

Definition

$\bar{R}^0_{A,ideal}$ is the intersection of all A-definable two-sided ideals of $\bar{R}$ of countable (equivalently, bounded) index. $\bar{R}^0_{A,ring}$ is the intersection of all A-definable subrings of $\bar{R}$ of countable index.

In contrast with the definable R, it may happen that $\bar{R}^0_{A,ring}$ does not exit, or that it exists but $\bar{R}^0_{A,ideal}$ does not.

Example

Let $M := (\mathbb{R}, +, \cdot, 0, 1)$ and $X := [-1, 1]$ which is clearly a 0-definable approximate subring (and here $R = \mathbb{R}$). Then $(\bar{R}, +)^0_M$ does not exist. Also, $\bar{R}^{00}_M = (\bar{R}, +)^{00}_M = \bigcap_{n\in\omega} \bar{I}_n =: \mu$, where $I_n := [-\frac{1}{n}, \frac{1}{n}]$ and $\bar{I}_n$ is the interpretation of I_n in $\mathfrak{C}$ (i.e. μ is the subgroup of the infinitesimals of $\bar{R}$), and $\bar{R}/\bar{R}^{00}_M$ is isomorphic to $\mathbb{R}$ as a topological ring, so it is not totally disconnected.

Example

Let $M := \mathbb{F}_p((t))$ be equipped with the full structure. Let X be the additive subgroup consisting of the series of the form $\sum_{i=-1}^{\infty} a_i t^i$. This is clearly a 0-definable approximate subring, and $R := \langle X \rangle = \mathbb{F}_p((t))$. Then $\bar{R}^0_{M,ideal}$ does not exist, while $\bar{R}^0_{M,ring}$ does exist. The ring $\bar{R}/\bar{R}^{00}_M$ is totally disconnected but does not have a basis of neighborhoods of 0 consisting of open ideals.

The last example shows that the counterpart of Thm. 1 for $\bar{R} = \langle \bar{X} \rangle$ (with "finite index" replaced by "countable index") fails. Also, the counterpart of Thm. 2(1) fails for $\bar{R}$.

Question

Suppose $(\bar{R}, +)^0_A$ exists. Is it true that $(\bar{R}, +)^0_A + \bar{R}(\bar{R}, +)^0_A$ is a subgroup of $(\bar{R}, +)$?

Proposition

Assume that $\bar{R}$ is of positive characteristic. Then $(\bar{R},+)^0_A$ exists and coincides with $(\bar{R},+)^{00}_A$. Thus, $(\bar{R},+)^0_A + \bar{R}(\bar{R},+)^0_A = \bar{R}^{000}_A$ (is a subgroup); if also $R \subseteq \mathrm{dcl}(A)$, then $(\bar{R},+)^0_A + \bar{R}(\bar{R},+)^0_A = \bar{R}^{00}_A$.