Rough approximate subgroups

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Approximate subgroups

Definition

Let K be an integer. A symmetric subset $1 \in A = A^{-1}$ of a group G is a K*-approximate subgroup* if there is a finite set E of size K such that $A^2 \subseteq EA$. If K is irrelevant, it is dropped.

- If A is finite, it is $|A|$-approximate, so the notion is only interesting for $K \ll |A|$.
- $K = 1$ iff A is a subgroup.
- If $K \leq 2$ and A is K-approximate, it is very close to a finite union of cosets.
- Breuillard, Green and Tao have classified finite approximate subgroups: asymptotically, up to commensurability and modulo a finite subgroup, they are nilprogressions generating a nilpotent subgroup.

Finite History

Abelian groups

- Freyman 1966, Ruzsa 1994: Complete classification of sets of small doubling $|A^2| \leq K\,|A|$ in torsion-free abelian groups. Weaker than approximate subgroup!
- Gowers 1998, 2001: Used Freyman's Theorem for a new proof of Szemerédi's theorem on arithmetic progressions in dense sets of integers.
- Green-Ruzsa 2007: Classification of sets of small doubling in general abelian groups.
- Bourgain-Katz-Tao 2004: Sum-Product Theorem (a subset of $\mathbb{F}_p$ cannot have simultaneous additive and multiplicative small doubling).

Finite History

Non-abelian groups

- Helfgott 2008: If A generates $SL_2(\mathbb{F}_p)$ and $|A^2| \leq c|A|^{1+\varepsilon}$, it is close to the full group. Weaker than small doubling!
- Tao 2008: Extension to non-commutative groups, definition of approximate subgroups.
- Tao 2010 (soluble), Safin 2011, Razborov 2014 (free), Breuillard-Green 2011, 11 & 12 (torsion-free nilpotent, soluble linear, unitary), Gill-Helfgott 2014 (soluble linear), Pyber-Szabó 2016 (finite simple Lie), Breuillard-Green-Tao 2011 (linear).
- Hrushovski 2012: Lie Model Theorem.
- Breuillard-Green-Tao 2012: Classification.

Applications

- Expander graphs,
- Groups of polynomial growth (Gromov's Theorem),
- Sieve theory,
- Additive combinatorics,
- Differential geometry,
- Random walks.

Tools

- Combinatorics,
- Convex geometry,
- Group theory,
- Representation theory,
- Harmonic analysis,
- Probability theory,
- Algebraic group theory,
- Model theory.

The Lie Model Theorem

Definition

A structure $\mathfrak{M}$ is *definably amenable* if there is a finitely additive measure μ on the algebra of definable subsets of *M* which is not concentrated on finitely many types. If $X \subseteq M$ is definable, the measure is *normalized* at *X* if $\mu(X) = 1$.

- We do not require that all definable subsets of *M* have finite measure (i.e. the value ∞ is allowed).
- If $\mathfrak{M}$ is a group, we usually require the measure to be left invariant. In particular this is true in this talk.
- We do not *a priori* require the measure to be automorphism invariant, or definable.
- We usually do not need the measure to exist on all definable subsets of *M*, but just on some suitable subalgebra.

Theorem

Let G be definably amenable normalized at a definable approximate subgroup $A \subseteq G$. Then there is a type-definable subgroup $N \subseteq A^4$ normalized by A and of bounded index in $\langle A \rangle$.

This was shown first by Hrushovski in the more general context of *near-subgroups* (existence of an S1-ideal of negligible sets) using a *stabilizer theorem*, then combinatorially by Sanders for finite approximate subgroups, generalized by Massicot-W. to the definably amenable case, and fine-tuned by Krupiński-Pillay and Hrushovski-Krupiński-Pillay with applications to connected components.

Corollary (Lie Model Theorem)

Under the same hypotheses, there is a $\bigvee$-definable group H commensurable with $\langle A \rangle$, a normal type-definable subgroup $K \leq H$ commensurable with N, a finite-dimensional Lie group L, and a homomorphism $\phi : H \to L$ with kernel K and dense image, such that for any $F \subseteq F' \subseteq L$ with F compact and F' open there is a definable $D \subseteq H$ with $\phi^{-1}[F] \subseteq D \subseteq \phi^{-1}[F']$. Any such D is commensurable with A^2.

To deduce the Corollary, just note that $\langle A \rangle / N$ is a locally compact group in the logic topology, where proper subsets are closed iff their pre-image is type-definable. The rest follows from the theory of locally compact groups.

And beyond

Massicot-W. have recklessly conjectured that definable amenability is not needed in the Lie Model Theorem. Counter-examples were given by Breuillard, Hrushovski, and Hrushovski-Krupiński-Pillay. However, Hrushovski has shown that it is true if we just ask for ϕ to be a *quasi-homomorphism*.

Theorem

Let $A \subseteq G$ be an approximate subgroup. Then there is a locally compact group L, a compact normal subset $\Delta \subseteq L$ and a quasi-homomorphism $\phi : G \to L$ with compact error set Δ such that

1. *For any compact $C \subseteq L$ there is n such that $\phi^{-1}[C] \subseteq A^n$.*
2. *For any n there is a compact $C \subseteq L$ such that $A^n \subseteq \phi^{-1}[C]$.*
3. $\phi^{-1}[\Delta] \subseteq A^{12}$.
4. *If $C_0, C_1 \subseteq L$ are compact with $\Delta^2 C_0 \cap \Delta^2 C_1 = \emptyset$, then there are disjoint definable $D_0, D_1 \subseteq A^n$ for some n with $\phi^{-1}[C_i] \subseteq D_i$ for $i = 1, 2$.*

Metric approximate subgroups

In his thesis under the direction of Hrushovski, Arturo Rodríguez Fanlo studies *metric approximate subgroups*, i.e. symmetric subsets A of some metric group G satisfying $A^2 \subseteq EAT$, where E is a finite set and T is the closed ball of infinitesimals. More precisely, rather than the approximate subgroup condition, he uses metric entropies N_{r_i}, where

$$N_{r_i}(X) = \text{the maximal number of } r_i\text{-separated points in } X$$

as proposed by Tao, and such that $r_{i+1} \leq r_i/2$. If

$$N_{r_i}(A^9) \leq k_i \cdot N_{9r_i}(A)$$

he obtains an S1-ideal of negligible sets making A into a near-subgroup, giving rise to a type-definable subgroup H; for k_i constant this yields a Lie model theorem for the approximate subgroup A^2.

The method of Hrushovski and Rodríguez Fanlo use a serious and non-canonical expansion of the language. Here, we shall use the Sanders-Massicot-W. approach to directly obtain H.

Instead of the normal ball of infinitesimals we shall consider an A-invariant type-definable subgroup, and metric entropy will be replaced by certain left-invariant outer measures.

T-rough measures

Let G be a group, $1 \in T \subseteq G$ a symmetric subset, and $\mathcal{A}$ a boolean algebra of definable subsets of G such that $XT \in \mathcal{A}$ and $gX \in \mathcal{A}$ for all $X \in \mathcal{A}$ and $g \in G$.

Definition

An *T-rough measure* on $\mathcal{A}$ is a left invariant outer measure $\mu : \mathcal{A} \to [0, \infty]$ which is finitely subadditive and additive modulo T:

$$\mu(X \cup Y) = \mu(X) + \mu(Y) \text{ whenever } X, Y \in \mathcal{A} \text{ with } X \cap YT = \emptyset.$$

For $\mathcal{A}$-type-definable Y put $\mu(Y) = \inf\{\mu(X) : Y \subseteq X \in \mathcal{A}\}$.

Lemma

Let μ be an T-rough measure on $\mathcal{A}$ and $X_0, \ldots, X_n \in \mathcal{A}$. Then

$$\mu\Big(\bigcup_{i\leq n} X_i\Big) \geq \sum_{i\leq n} \mu(X_i) - \sum_{i<j\leq n} \min\{\mu(X_i \cap X_j T), \mu(X_i T \cap X_j)\}.$$

By taking limits, this also holds for $\mathcal{A}$-type-definable X_i.

Thickness

Definition

A definable symmetric subset X of G is *t-thick in A^m* if for any $t+1$ elements $g_0, \ldots, g_t$ in A^m there is $0 \leq i \neq j \leq t$ with $g_i^{-1} g_j \in X$. A type-definable subset of G is *thick in A^m* if every definable superset is t-thick for some $t < \omega$.

Note that if $B \subseteq G$ is t-thick in A^m, then t left translates of B cover A^m.

Lemma

A finite intersection of subsets of G which are t-thick in A^m is t'-thick in A^m, for some $t' < \omega$. An intersection of subsets thick in A^m is still thick.

Proof.

The first assertion is by Ramsey's Theorem. The second one follows. □

Lemma

Let μ be a T-rough measure on $\mathcal{A}$, and suppose $A \in \mathcal{A}$ with $0 < \mu(A^2) < \infty$. Let $t > 0$ be an integer and $B \subseteq A$ definable with $\mu(B) \geq 2\mu(A^2)/t$. Put

$$S(B) = \left\{ g \in A^2 : \min\{\mu(B \cap gBT), \mu(gB \cap BT)\} \geq \frac{2\mu(A^2)}{t^2} \right\}.$$

Then $S(B)$ is t-thick in A.

Note that $B \cap gBT$ or $gB \cap BT$ non-empty implies $g \in BTB^{-1}$.

Proof.

Otherwise find $(g_n : n \leq t)$ in X such that $g_n \notin \bigcup_{k<n} g_k S$ for all $n \leq t$. By the Lemma,

$$\mu(A^2) \geq \mu\Big(\bigcup_{i \leq t} g_i B\Big) > (t+1)\frac{2\mu(A^2)}{t} - \frac{(t+1)t}{2}\,\frac{2\mu(A^2)}{t^2}$$

$$= \frac{t+1}{t}\,\mu(A^2) > \mu(A^2), \qquad \text{a contradiction.} \quad \square$$

T-rough definable amenability

Fix $A \in \mathcal{A}$. Let $T = \bigcap_{i<\omega} T_i$ be a type-definable A-invariant subgroup, where each T_i is symmetric, $T_{i+1}^2 \subseteq T_i$ and $T_{i+1}^A \subseteq T_i$ for all $i < \omega$.
Suppose that $XT_i \in \mathcal{A}$ for all $i < \omega$ and $X \in \mathcal{A}$.

Definition

A is *T-rough definably amenable* if for all $i < \omega$ there is a left G-invariant T_i-rough measure μ_i on $\mathcal{A}$ normalised at A.

Theorem

Let $A \subseteq G$ be T-rough definably amenable, and $m, m', (K_i)_{i<\omega}$ integers with $\mu_i(A^{2m+m'}) \leq K_i$. Suppose $B = \bigcap_{i<\omega} B_i \subseteq A^m$ is type-definable and thick in A. Then there is a type-definable subset $S \subseteq B^2 \subseteq A^{2m}$ thick in A^m such that $S^{m'} \subseteq B^4 T$.

Consider the following definable conditions $P^t_{n,i}(X)$ on definable subsets $X \subseteq A^m$, for $n, i, t < \omega$:

- $P^t_{0,i}(X)$ if $X \neq \emptyset$.
- $P^t_{n+1,i}(X)$ if

$$S^t_{n,i}(X) = \left\{ g \in A^{2m} : P^{t^2}_{n,i}(X \cap gXT_i) \text{ and } P^{t^2}_{n,i}(X \cap g^{-1}XT_i) \right\}$$

  is t-thick in A^m.

Lemma

The $P^t_{n,i}(X)$ and $S^t_{n,i}(X)$ are monotone in n, t and X. □

By the Lemma and induction on n we have $P^t_{n,i}(X)$ whenever $X \subseteq A^m$ with $\mu_i(X) \geq 2K_i/t$. But $\mu_i(B_i) > 0$, as any finitely many translates of any B_i cover A. So for $t_i \geq 2K_i/\mu_i(B_i)$ we have $P^{t_i}_{n,i}(B_i)$ for all $n < \omega$. Thus $S_0 = \bigcap_{i,n<\omega} S^{t_i}_{n,i}(B_i)$ is thick in A^m.

Choose $\varepsilon_i < \mu_i(B_i)/m'K_i$, and start a sequences $(X_{i,n})_{n<\omega}$ with $X_{i,0} = B_i$ for all $i < \omega$. Then $X_0 = \bigcap_{i<\omega} X_{i,0} = B$.
If there are minimal $i_0 < \omega$, as well as $k_0 < m'$ and $g_0 \in S_0$ with

$$\begin{aligned}\mu_{i_0}((X_{i_0,0} \cap g_0 X_{i_0,0} T_{i_0}) T^{k_0}_{i_0} B_{i_0} T^{k_0}_{i_0} \cap A^{2m+m'}) \\ \leq (1 - \varepsilon_{i_0}) \mu_{i_0}(X_{i_0,0} T^{k_0}_{i_0} B_{i_0} T^{k_0}_{i_0} \cap A^{2m+m'}),\end{aligned}$$

define $X_{i,1} = X_{i,0} \cap g_0 X_{i,0} T_i$, so $X_1 = \bigcap_{i<\omega} X_{i,1} = X_0 \cap g_0 X_0 T$, and iterate. Otherwise, stop.

Note that $P^{t_i^{2^n}}_{n,i}(X_{i,n})$ holds for all $i, n < \omega$, and $\lim_{n\to\infty} i_n = \infty$.
Put $X = \bigcap_{n<\omega} X_n$, and $S = \bigcap_{n<\omega} S_n$, a type-definable subset of $B^2 T \cap A^{2m}$ thick in A^m.

If $n \gg 0$ then $i_n > i$, and for any $g \in S$ and $k < m'$ we have

$$\mu_i(X_{i,n}T_i^k B_i T_i^k \cap gX_{i,n}T_i^{k+1}B_i T_i^k \cap A^{2m+m'}) \geq \mu_i((X_{i,n} \cap gX_{i,n}T_i)T_i^k B_i T_i^k \cap A^{2m+m'}) \geq (1-\varepsilon_i)\mu_i(X_{i,n}T_i^k B_i T_i^k \cap A^{2m+m'}), \text{ whence}$$

$$\mu_i((X_{i,n}T_i^k B_i T_i^k \setminus gX_{i,n}T_i^{k+1}B_i T_i^{k+1}) \cap A^{2m+m'}) \leq \mu_i(X_{i,n}T_i^k B_i T_i^k \cap A^{2m+m'}) - \mu_i(X_{i,n}T_i^k B_i T_i^k \cap gX_{i,n}T_i^{k+1}B_i T_i^k \cap A^{2m+m'}) \leq (1-(1-\varepsilon_i))\,\mu_i(X_{i,n}T_i^k B_i T_i^k \cap A^{2m+m'}) \leq \varepsilon_i\,K_i.$$

If $g_0, \ldots, g_{m'-1} \in S$, write $g_{<\ell} := g_0 \cdots g_{\ell-1}$ for each $\ell < m'$. So

$$\mu_i((X_{i,n}B_i \setminus g_{<m'}X_{i,n}T_i^{m'}B_i T_i^{m'}) \cap A^{2m}) \leq \mu_i\left(\bigcup_{k<m'} g_{<k}(X_{i,n}T_i^k B_i T_i^k \setminus g_k X_{i,n}T_i^{k+1}B_i T_i^{k+1}) \cap A^{2m}\right) \leq \sum_{k<m'} \mu_i((X_{i,n}T_i^k B_i T_i^k \setminus g_k X_{i,n}T_i^{k+1}B_i T_i^{k+1}) \cap A^{2m+m'}) \leq m'\,\varepsilon_i\,K_i < \mu_i(B_i) \leq \mu_i(X_{i,n}B_i \cap A^{2m}).$$

In particular, $X_{i,n}B_i \cap g_{<m'}X_{i,n}T_i^{m'}B_iT_i^{m'} \neq \emptyset$.

By compactness, $XB \cap g_{<m'}XBT \neq \emptyset$ (since T is A-invariant).

Thus $S^{m'} \subseteq B^4T \subseteq A^{4m}T$. □

Theorem

Let $A \subseteq G$ be T-rough definably amenable and suppose that $\mu_j(A^m) < \infty$ for all $j, m < \omega$. Then there is $N \trianglelefteq \langle AT \rangle$ type-definable of bounded index such that $T \subseteq N \subseteq A^4T$.

Proof.

We iterate the Theorem with $m' = 8$, putting

$$A_0 := A \qquad \text{and} \qquad A_{n+1} := S \subseteq A_n^2 \subseteq A^{2^n} \text{ for } B := A_n.$$

Clearly, $H = \bigcap_{n<\omega} A_n^4T$ is a type-definable subgroup in A^4T thick in A^mT for all $m < \omega$. Thus the index of H in $\langle AT \rangle$ is bounded.

It follows that the intersection of all $\langle AT \rangle$-conjugates of H is a bounded intersection, whence a type-definable normal subgroup N in $\langle AT \rangle$ of bounded index. □

Definition

$A \subseteq G$ is a *T-rough K-approximate subgroup* if $A^2 \subseteq EAT$ for some finite E of size K.

Lemma (Ruzsa)

Let $A \subseteq G$ be T-rough definably amenable, and suppose $\mu_i(A^4) < K$ for all i. Then A^2 is a T-rough K^2-approximate subgroup.

Lemma

Let $A \subseteq G$ be a T-rough definably amenable T-rough approximate subgroup with $\mu_i(AT) < \infty$ for all $i < \omega$. Then $\mu_i(A^m) < \infty$ for all $i, m < \omega$.

Corollary

Let $A \subseteq G$ be T-rough definably amenable T-rough approximate subgroup with $\mu_i(AT) < \infty$ for all $i < \omega$. Then $\langle A \rangle$ has a Lie model.

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Thank you !