Model Theory of Modules over Prüfer Domains via their Value Groups

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Bézout domains: PIDs, ring of algebraic integers, ring of complex entire functions.

Prüfer domains: Bézout domains, Dedekind domains, the ring of integer valued polynomials (with coefficients in $\mathbb{Q}$).

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The Baur-Monk Theorem

Every formula in the language of R-modules is equivalent to a boolean combination of pp-formulae and sentences

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where $n \in \mathbb{N}$ and φ, ψ are pp-1-formulae such that $\varphi \geq \psi$.

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We write $\Gamma(R)^+_\infty$ for the positive cone of $\Gamma(R)$ with a greatest element ∞ adjoined.

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Theorem (G.)

Let R, S be Prüfer domains. If $\Gamma(R) \cong \Gamma(S)$ then there exists a lattice isomorphism $\lambda : \mathrm{pp}_S^1 \to \mathrm{pp}_R^1$ and a homeomorphism of topological spaces $\rho : Zg_R \to Zg_S$ such that for all $\varphi, \psi \in \mathrm{pp}_S^1$, $(\lambda\varphi/\lambda\psi) = \rho^{-1}(\varphi/\psi)$.

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- Let $\sim_0$ be the trivial congruence relation and let $L := L_0$.
- Let $L_{\alpha+1} := L_\alpha / \sim$ and $\sim_{\alpha+1}$ be the congruence kernel of the quotient map from L to $L_{\alpha+1}$.

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- R is a Prüfer domain [G.].

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Corollary (Puninski-Toffalori, G.)

Let R be a Prüfer domain. The breadth of pp^1_R is equal to the m-dimension of $\Gamma(R)^+_\infty$.

Proof?

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Proposition (Puninski)

If R *is arithmetical then the m-dimension of* pp_R^1 *is equal to the Cantor-Bendixson rank of* Zg_R.

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