Interpretable groups and fields in various valued fields

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What is the project?

Let $\mathcal{K} = (K, +, \cdot, v, \cdots)$ be an expansion of a valued field. Under additional assumptions, we want to study definable and interpretable groups and fields in $\mathcal{K}$.

What is meant here by “Interpretable”?

A group $(G; \cdot)$ is *interpretable* in $\mathcal{K}$ if there are
(i) definable set $X \subseteq K^n$ and a definable equivalence relation E on X, and
(ii) a function $M : (X/E)^2 \to X/E$ such that
$(G; \cdot) \simeq (X/E; M)$, and the preimage under π of $Graph(M)$ is a definable subset of X^3.
An interpretable field is similarly defined.

The plan of this tutorial

Talk I. 3 settings and 4 distinguished sorts-the Closed Ball Property

Talk II. Dimension, rank and the Independent Neighborhood Property

Talk III. Infinitesimal subgroups of an interpretable group

Valued Fields

Definition

A (non-archimedean) valuation on a field K is a map $v : K \to \Gamma \cup \{\infty\}$, for Γ an ordered abelian group, satisfying:

1. $v(x) = \infty \Leftrightarrow x = 0$.
2. $v(x \cdot y) = v(x) + v(y)$ (a homomorphism $: K^\times \to (\Gamma, +)$).
3. $v(x + y) \geqslant \min\{v(x), v(y)\}$

Notation

For $\gamma \in \Gamma$, $a \in K$,

$B_{>\gamma}(a) = \{x \in K : v(x - a) > \gamma\}$ $\quad B_{\geqslant\gamma}(a) = \{x \in K : v(x - a) \geqslant \gamma\}$.

$\mathcal{O} = B_{\geqslant 0}$ the valuation ring

$\mathbf{m} = B_{>0} \subseteq \mathcal{O}$ the maximal ideal

$\mathbf{k} = \mathcal{O}/\mathbf{m}$ the residue field

Some basic examples of valued fields

• $K = k((t))$ the field of Laurent series of some field k
Let $v(\sum_{k\in\mathbb{Z}} a_k t^k) = min\{k : a_k \neq 0\}$. Here $\Gamma = \mathbb{Z}$, and $\mathbf{k} = k$.

• (p**-adic valuation**) Consider $\mathbb{Q}$, **fix** p **prime**, and define $v(a/b) = n$ if $a/b = p^n(a'/b')$ with $gcd(a', b') = 1$. Here $\Gamma = \mathbb{Z}$ and $\mathbf{k} = \mathbb{F}_p$,
The valuation v endows $\mathbb{Q}$ with a metric $d(x, y) = p^{-v(x-y)} \in \mathbb{R}$.

• **The** p**-adic field,** $\mathbb{Q}_p =$ the completion of $\mathbb{Q}$ with respect to the p-adic metric. The elements can be written as

$$\sum_{n\geqslant m} a_k p^k,$$

with $m \in \mathbb{Z}$, $0 \leqslant a_k \leqslant p - 1$, and addition with “carry over”.
We have $\Gamma = \mathbb{Z}$ and $\mathbf{k} = \mathbb{F}_p$.

• **Finite extensions of** $\mathbb{Q}_p$ admit valuations extending the p-adic one.
Here $\Gamma \cong \mathbb{Z}$, $\mathbf{k} =$ finite extension of $\mathbb{F}_p$.

The 3 main settings, with logic

We start with $\mathcal{K} = (K; +, \cdot, v)$ a valued field in the signature of $+, \cdot, v$, equivalently $\mathcal{K} = (K; +, \cdot, \mathcal{O})$, where $\mathcal{O}$ the valuation ring.

- **pCF, p-adically closed fields.**

$\mathcal{K}$ is elementarily equivalent to finite extensions of $\mathbb{Q}_p$.

We shall also consider "P-minimal expansions" (to be defined).

- **RCVF, Real closed valued fields.**

$\mathcal{K} = (K; +, \cdot, <, \mathcal{O})$, where K is a real closed field and $\mathcal{O} \subseteq K$ a nontrivial convex (valuation) subring (containing 1!).

We shall also consider "T-convex, power bounded expansions" (t.b.d).

- **$\mathrm{ACVF}_{0,0}$, algebraically closed valued fields.**

$\mathcal{K} = (K; +, \cdot, \mathcal{O})$ be an algebraically closed field, $char(K) = 0$, $\mathcal{O}$ a nontrivial valuation ring, $char(\mathbf{k}) = \mathbf{0}$.

We shall also consider "V-minimal expansions" (not to be defined....).

P-minimal expansions of p-adically closed fields

Fix prime p, and assume that K is a p-adically closed field.

Main properties

- The residue field **k** is a finite extension of $\mathbb{F}_p$.
- The ordered value group Γ (with the induced structure) is $\equiv$ to $(\mathbb{Z}; <, +)$, namely a $\mathbb{Z}$-group. (Note, closed balls in K are open balls).
- (Macintyre) Every definable subset of K is a boolean combination of singletons, balls and cosets of $P_n = \{x \in K^* : \exists y \in K \; y^n = x\}$.
- (v. d. Dries) $\mathcal{K}$ has definable Skolem function (for the home sort K !).

Definition (Haskell-Macpherson)

An expansion $\mathcal{K} = (K; +, \cdot, v, \cdots)$ is *P-minimal* if for every $\mathcal{K}' \equiv \mathcal{K}$,
(i) $\Gamma_{K'}$ is a $\mathbb{Z}$-group,
(ii) every definable subset of K' is definable in the field language.

v.d.Dries-Haskell-Macpherson: $Th(\mathbb{Q}_p, \mathcal{L}_{an})$ is P-minimal.

The closed ball property-the P-minimal case

Theorem

Assume that $\mathcal{K}$ is P-minimal.
If $X \subseteq K$ is definable and intersects infinitely many closed 0-balls then X contains a closed ball of radius < 0.

In fact, for every $k \in \mathbb{N}$, X contains a ball of radius $< -k$.

A preliminary observation

Every ball with (valuative) radius in $\mathbb{Z}$ intersects only finitely many closed 0-balls. Indeed, this follows directly from the fact that the residue field is finite.

Proof of closed ball prop. (thanks to D. Macpherson)

(For simplicity of presentation, assume saturation)

- Using P-minimality (and $+$-translation), we may assume $X = \{x \in K : \gamma_1 < v(x) < \gamma_2 \text{ and } \lambda x \in P_n\}$, for $\gamma_1, \gamma_2 \in \Gamma \cup \{\pm\infty\}$, and $\lambda \in K$, the intersection of an annulus with a coset of P_n^*.
- By the observation, $\gamma_1 < \mathbb{Z}$. Hence, there exists $x_0 \in X$ with $v(x_0) = \gamma_0 < \mathbb{Z}$. So, $\gamma_0 < \gamma_2$.
- **We claim that** $B_{\geqslant \gamma_0/2}(x_0) \subseteq X$:
- If $x \in B_{\geqslant \gamma_0/2}(x_0)$ then $\gamma_1 < v(x) = v(x_0) < \gamma_2$. Enough to see $x^{-1}x_0 \in P_n^*$ (hence also $\lambda x \in P_n^*$):
- Let $f(Y) = Y^n - x^{-1}x_0 \in \mathcal{O}[Y]$.
- $v(f(1)) = v(1 - x^{-1}x_0) = v((x - x_0)/x) = v(x - x_0) - v(x) = -\gamma_0/2 > \mathbb{N}$.
- $v(f'(1)) = v(n) \in \mathbb{N}$, so $v(f(1)) > 2v(f'(1))$.
- By Hensel's lemma, $f(Y)$ has a root in $\mathcal{O}$, hence $x^{-1}x_0 \in P_n$. □

T-convex real closed valued fields

Let K be real closed, $\mathcal{M} = (K; <, +, \cdot, \cdots)$ o-minimal, polynomially bounded (or "power bounded").
Let $\mathcal{O} \subsetneq K$ be a convex ring, closed under all $\emptyset$-definable continuous functions $f : \mathcal{O} \to K$ (T-convex). In particular, $\mathcal{O}$ is a valuation ring.

Theorem (v.d. Dries-Lewenberg, v.d. Dries)

The expansion $\mathcal{K} = (K; <, +, \cdot, \mathcal{O}, \cdots,)$ is a real closed valued field.

• (no need for "power bounded") $(K :<, \cdots)$ is weakly o-minimal and has definable Skolem functions, after naming $a >> \mathcal{O}$.

• The residue field **k**, with induced structure, is an o-minimal structure, elementarily equivalent to $\mathcal{M}$.

• The value group Γ, with the induced structure, is an ordered vector space, over "the field of powers".

• (Tyne) Every definable subset of K is a boolean combination of balls and intervals.

The closed balls property-the T-convex case

Theorem

Assume $X \subseteq K$ is a definable set, which intersects infinitely many closed balls of radius 0.
Then X contains at least one closed ball of radius < 0.
Note that this fails for open balls.

Proof:

- By weak o-minimality, X is a finite union of convex subsets of K so we may assume that X is convex.
- Balls are convex sets. Hence, X must contain infinitely many closed 0-balls.
- (Because balls are closed) There are $x_1 < x_2$ in X with $v(x_1 - x_2) = \gamma < 0$.
- The ball $B_{\geqslant \gamma/2}(\frac{x_1+x_2}{2})$ is contained in the interval $(x_1, x_2) \subseteq X$. □

C-minimal expansions of ACVF_0.

Definition

An expansion $\mathcal{K} = (K; +, \cdot, v \cdots)$ of algebraically closed valued field of char 0 is *C-minimal* if in every $\mathcal{K}' \equiv \mathcal{K}$, every definable subset of K' is quantifier-free definable in the valued field language.

Theorems

1. (Robinson) Algebraically closed valued fields are C-minimal.
2. (Haskell-Macpherson) If $\mathcal{K}$ is C-minimal then

   • Γ, with induced structure, is an ordered divisible abelian group (o-minimal).

   • The residue field **k**, with induces structure, is a strongly minimal expansion of an algebraically closed field.

The closed ball property, the C-minimal case

Theorem

If $\mathcal{K}$ is C-minimal and $X \subseteq K$ definable and intersects infinitely many closed 0-balls then X contains a closed ball of radius $\gamma < 0$.

Below, *a maximal sub-ball of* X is a ball $B \subseteq X$ which is not properly contained in any other ball in X.

Conclusion in all settings from the closed ball property

If $X \subseteq K$ is definable then there are at most finitely many maximal closed sub-balls of X of every fixed radius in X.

Elimination of imaginaries

The “correct” model theoretic machinery isolates some basic sorts in $\mathcal{M}^{eq}$ and reduces analysis of **all** definable quotients to these sorts.

Sometimes these sorts are not needed (ACF, o-minimal expansions of groups, DCF_0)

Some theorems on elimination of imaginaries in valued fields

- (Haskell-Hrushovski-Macpherson) Algebraically closed valued fields eliminate imaginaries when we add “geometric sorts”.
- (Mellor) Real closed valued fields eliminate imaginaries with the “geometric sorts”.
- (Hrushovski, Martin, Rideau-Kikuchi) p-adically closed fields eliminate imaginaries in appropriate language.

The 4 distinguished sorts

Difficulties in applications of the EI results

The geometric sorts are complex, of unbounded dimension, it is not simple to understand definable quotients through them.

A lazy way out

Instead, we drop the hope to **fully** analyze interpretable groups and fields via the special sorts. We focus our attention on 4 “one dimensional distinguished sorts” , and analyze groups and fields **locally** through these.

The 4 distinguished sorts

- K
- $\mathcal{O}/\mathbf{m} = \mathbf{k}$, open 0-balls.
- $K/\mathcal{O}$, the closed 0-balls.
- $K^*/\mathcal{O}^* = \Gamma$.

Reduction to the 4 distinguished sorts

Theorem: Reduction to the distinguished sorts

Let $\mathcal{K} = (K; +, \cdot, \cdots)$ be an ω-saturated
(i) C-minimal expansion of ACVF, or
(ii) T-convex expansion of RCVF, or
(iii) P-minimal expansion of p-adically closed field.

If X/E is a definable infinite quotient, $X \subseteq K^n$, then there exists an infinite definable $Y \subseteq X/E$, and a definable finite-to-finite correspondence between Y and a subset of K, $\mathbf{k}$, $K/\mathcal{O}$, or Γ.

Proof
Step 1: (totally general) There exists a finite-to-finite definable correspondence between an infinite subsets of X/E and K/E_1 for some definable equivalence relation E_1 on K:
This is discrete mathematics (no connection to fields). Exercise.

Step 2

Note: all value groups have Definable Choice

We have X/E, for $X \subseteq K$.

- For each E-class $C \in X/E$, let $S_{C,max}$ be the set of all maximal balls (**open, closed, or singletons**) inside C. It is definable.
- Using Definable Choice, let $\gamma_C \in \Gamma \cup \{\infty\}$ be one of the radii of balls in $S_{C,max}$, We may assume that for all $C \in X/E$, all $b \in S_{max,C}$ have the same radius $\gamma(C)$.
- **Case 1** The map $C \mapsto \gamma(C)$ is finite-to-one. Then $X/E \sim \Gamma$.
- **Case 2** There is $\gamma_0 \in \Gamma$ with $\gamma^{-1}(\gamma_0) \subseteq X/E$ infinite. This is the new X/E. Now, all maximal balls in all classes have the same radius γ_0.

Proof continues

So we now assume that for every class $C \in X/E$, every maximal ball has the same radius γ_0.

- If $\gamma_0 = \infty$ then every C is a union of isolated points so finite. $X/E \sim K$.
- So assume $\gamma_0 \in \Gamma$. By the closed ball property, each C intersects at most finitely many closed balls of radius γ_0.
- So, we have a 1-finite map from X/E into " the closed balls of radius γ_0" $\sim K/\mathcal{O}$.
- **Case 2.1** Each closed ball of radius γ_0 intersects at most finitely many C's. In this case $X/E \sim X/\mathcal{O}$.
- **Case 2.2** Some closed ball b_0 of radius γ_0 intersects ∞-many classes C. WMA all balls in $S_{max,C}$ are contained in b_0, so balls in $S_{max,C}$ must be open....... Working in $B_{\geqslant\gamma_0}/B_{>\gamma_0} \sim \mathbf{k}$, we get $X/E \sim \mathbf{k}$. □

Dimension and rank

In all settings (ACVF, RCVF, pCF), **in the home sort** K, *acl* satisfies Steinitz Exchange and $\exists^\infty$ is eliminated, thus these are **geometric structures**-we have a good notion of dimension with additivity:

$$\forall \bar{a}, \bar{b} \in K, \text{ and } A, \dim(\bar{a}, \bar{b}/A) = \dim(\bar{a}/\bar{b}A) + \dim(\bar{b}/A).$$

However, what is a good dimension for definable quotients?

Option 1

(Gagelman 2004): If D a geometric structure then one can extend dimension to D^{eq} and maintain additivity.
Problem: The sorts Γ, $K/\mathcal{O}$ and $\mathbf{k}$ are all 0-dimensional (because the equivalence classes are 1-dimensional), so we “lose them”.

Option 2: dp-rank (Usvyatsov, 2009)

The notion of dp-rk is defined for any tuple and definable set in $\mathcal{M}^{eq}$. (we omit the definition). We have: $\text{dp-rk}(\mathcal{M}) < \infty$ iff $Th(\mathcal{M})$ is NIP.

Basic properties

(1) $\text{dp-rk}(X) = 0$ iff X is finite.
(2) If $f : X \to Y$ is definable then $\text{dp-rk}(Y) \leqslant \text{dp-rk}(X)$.
(3) $\text{dp-rk}(X \times Y) = \text{dp-rk}(X) + \text{dp-rk}(Y)$.
(4)$\text{dp-rk}(X \cup Y) = \max\{\text{dp-rk}(X), \text{dp-rk}(Y)\}$.
(5) (*Subadditivity*)(Kaplan-Onshuus-Usvyatsov 2011)
$\text{dp-rk}(a, b/A) \leqslant \text{dp-rk}(a/bA) + \text{dp-rk}(b/A)$

The rank of the distinguished sorts is 1

- In all of our cases, $\text{dp-rk}(K) = 1$ (K is **dp-minimal**).
- $\text{dp-rk}(K/\mathcal{O}) = \text{dp-rk}(\Gamma) = 1$ (infinite image of an infinite subset of K).
- If $\mathbf{k}$ infinite then $\text{dp-rk}(\mathbf{k}) = \mathbf{1}$.

An important example $K/\mathcal{O}$

Take $\mathcal{K} = (K; <, +, \cdot, \mathcal{O})$ an RCVF.

- Since $\mathcal{O}$ is a convex subgroup, the sort $(K/\mathcal{O}; +, <)$ is a linearly ordered, weakly o-minimal (non-pure!) group.
- Fix $\alpha \in \mathbf{m}$ ($v(\alpha) > 0$). For $x - y \in \mathcal{O}$, $\alpha \cdot x - \alpha \cdot y \in \mathcal{O}$, Thus $x \mapsto \alpha \cdot x$ **descends** to an endomorphism $\alpha^* : K/\mathcal{O} \to K/\mathcal{O}$.
- $ker(\alpha^*) = \{x + \mathcal{O} : \alpha \cdot x \in \mathcal{O}\} = \{x + \mathcal{O} : v(x) \geqslant -v(\alpha)\}$, so α^* is locally constant. $\Rightarrow$ $K/\mathcal{O}$ is **not** a geometric structure-No Exchange:Take $a \in K$ generic over α, $b = \alpha^*(a)$. Then $b \in acl(a, \alpha) \setminus acl(\alpha)$ but $a \notin acl(b, \alpha)$.

Although there is no Exchange , $\text{dp-rk}(\bar{b}/A) = \dim_{acl}(\bar{b}/A)$!!!

$\dim_{acl}(a_1, a_2, \ldots, a_n) =$ maximal size of an *acl*-independent sub-tuple.

Fact: dp-rank and algebraic closure

In all distinguished sorts in our settings, $\text{dp-rk} = \dim_{acl}$

Simon-Walsberg uniformities (2015)

Definition

Let D be a dp-minimal expansion of a definable Hausdorff uniformity (e.g. topological group). We call it a *an SW-uniformity* if

1. D has no isolated points.
2. Every infinite definable $X \subseteq D$ has non-empty interior.

Examples

• O-minimal and weakly o-minimal structures (dense linear order).
• (Jahnke-Simon-Walsberg, Johnson) Every dp-minimal expansion of (nontrivially) valued field is an sw uniformity.
• in pCF, $(\Gamma, <, +)$ and $K/\mathcal{O}$ are not SW uniformities!

Important properties of SW uniformities (Simon-Walsberg)

(1) $\mathrm{dp\text{-}rk} = \dim_{acl}$. (2) Definable functions are continuous at generic points. (3) $\mathrm{dp\text{-}rk}\ \mathit{Frontier}(X) < \mathrm{dp\text{-}rk}(X)$.

The Independent Neighborhood property, version 1

A topological version

Let D be an SW uniformity. Let $X \subseteq D^n$ be a definable set (over any parameters), $a \in Int(X)$, and A any parameter set.
Then there exists $C \supseteq A$ and a C-definable open $U \subseteq X$, $a \in U$, such that $\mathrm{dp\text{-}rk}(a/C) = \mathrm{dp\text{-}rk}(a/A)$. Moreover, $U = U_1 \times \cdots U_n \subseteq D^n$.

Example

$\mathcal{M}$ o-minimal, $X \subseteq M^2$, $\langle a_1, a_2 \rangle \in Int(X)$. Then we can find open intervals $(b_1, b_2) \ni a_1$, $(b_3, b_4) \ni a_2$, such that $(b_1, b_2) \times (b_3, b_4) \subseteq X$ and $\dim(a/b_1 \cdots b_4) = \dim(a/A)$.

A simple application

Assume that D is an SW uniformity $Y \subseteq D^n$ definable and $f : Y \to Z$ definable finite-to-one, all defined over A.
Then, for every generic $x_0 \in Y$ over A, there exists $C \supseteq A$, such that $\mathrm{dp\text{-}rk}(x_0/C) = \mathrm{dp\text{-}rk}(x_0/A)$, and a C-definable open $U \ni x_0$ such that $f \restriction U \cap Y$ is injective.

Proof First choose any open $X \ni x_0$ such that $f^{-1}(f(x_0)) \cap X = \{x_0\}$. Then apply the IN property to replace X by $U \subseteq X$. □

The Independent Neighborhood property, version 2

The case of $\mathbb{Z}$ and $\mathbb{Q}_p/\mathbb{Z}_p$

What to do with distinguished sorts that are not SW uniformities?

E.g. Γ and $K/\mathcal{O}$ in the p-adic case: the natural topology on $\mathbb{Z}$ and $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete.

a non-topological version, for $\mathrm{dp\text{-}rk}(D) = 1$

Property (IN) For A, B any parameter sets, $X \subseteq D^n$ B-definable, and $a \in X$, such that $\mathrm{dp\text{-}rk}(a/B) = n = \mathrm{dp\text{-}rk}(a/A)$.

Then there exists $C \supseteq A$ and a C-definable $U \subseteq X$ such that $a \in U$ and $\mathrm{dp\text{-}rk}(a/C) = n$. Moreover, $U = U_1 \times \cdots \times U_n \subseteq D^n$.

Examples of Property (IN)

- Property (IN) fails for, say, ACF.
- Property (IN) holds for SW uniformities, and for Γ and $K/\mathcal{O}$ in pCF.
- **Question/Conjecture:** Is (IN) true in every distal (dp-minimal) structure?

An application of property (IN)

Theorem (A filter base)

Assume that $\operatorname{dp-rk}(D) = 1$ satisfies (IN), and $a \in D^n$, with $\operatorname{dp-rk}(a/\varnothing) = n$. Consider the global type:

$$\nu(a) = \{X \subseteq D^n \text{ definable over } A \subseteq M : a \in X, \text{ and } \operatorname{dp-rk}(a/A) = n\}.$$

Then (1) $D^n \in \nu(a)$. (2) For every $X, Y \in \nu(a)$ there is $Z \in \nu(a)$ such that $Z \subseteq X \cap Y$. In particular, $\nu(a)$ is consistent of rank n.

Proof

- Assume X, Y are definable over A, B, respectively.
- By (IN), there are $C \supset A$, and $Y' \subseteq Y$ which is C-definable, $a \in Y'$, such that $\operatorname{dp-rk}(a/C) = n$.
- Let $Z := X \cap Y' \subseteq X \cap Y$. Then Z is definable over C, $a \in Z$, and $\operatorname{dp-rk}(a/C) = n$, so $Z \in \nu(a)$. □

Summary so far

Which distinguished sorts are SW uniformities?

- In the ACVF case, K, $K/\mathcal{O}$ and Γ are SW uniformities.
(**k** is stable, so no definable Hausdorff topology).
- In the RCVF case, K, $K/\mathcal{O}$, Γ, and **k** are SW uniformities.
- in the pCF case, only K is an SW uniformity. $K/\mathcal{O}$ and Γ are not.

Which distinguished sorts satisfy property (IN)?

- All SW uniformities.
- **Proposition**: In the pCF case, $K/\mathcal{O}$ and Γ satisfy (IN)
(**k** is finite)
- In the ACVF case, **k** does not satisfy (IN).

Interpretable groups

Let G be an interpretable group in one of our settings.

We saw that there is a definable correspondence between an infinite $X \subseteq G$ and one of the distinguished sorts $D = K, K/\mathcal{O}, \mathbf{k}, \Gamma$.

With a bit more work, there is a finite-to-one map $f : X \subseteq G \to D^n$.

Example of $\mathrm{dp}\text{-}\mathrm{rk}(G) = 1$

- $G = K/\mathcal{O}$. In the ACVF and RCVF, G is a divisible abelian group.

In $\mathbb{Q}_p$: $G = \mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Z}(p^\infty)$ is the Prüfer group .

- $(G = RV)$

$1 \to \mathbf{k}^* \longrightarrow RV = K^*/(1+\mathbf{m}) \longrightarrow \Gamma \to 0$. It contains a copy of $\mathbf{k}^*$. When $\mathbf{k}$ is finite (pCF), then the angular component is definable, and then the set $\{x(1+\mathbf{m}) \in RV : ac(x) = 1\}$ in bijection with Γ.

- $G = (K/\mathbf{m}, +)$. It is dp-minimal and contains a copy of $(\mathbf{k}, +)$.

Another example

- Let $G = (K, +) \times (K/\mathcal{O}, +)$.
- $\mathrm{dp\text{-}rk}(G) = 2$.
- G has definable sets in correspondence (even bijection) with both K and $K/\mathcal{O}$.
- Clearly, every $K \times \{y\}$ is such a set. Also, the graph of $\pi : K \to K/\mathcal{O}$

is a definable subset (subgroup) of G in bijection with K.

- The only subsets of G in finite-to-finite correspondence (bijection) with $K/\mathcal{O}$ are of the form $\{a\} \times K/\mathcal{O}$.

Interpretable groups-the main theorem

Main theorem

Let $\mathcal{K} = (K; +, \cdot, v, \cdots)$ be either
(i) a (V-minimal expansion of) $\text{ACVF}_{0,0}$, or
(ii) a polynomially (power) bounded T-convex expansion of RCVF, or
(iii) a pCF.

Let G be an infinite group interpretable in $\mathcal{K}$.

Then, after possibly replacing G with G/H for H finite normal, there exists an infinite type-definable subgroup $\nu \subseteq G$ such that:

1. ν is definably isomorphic to a type definable group in K *or* in $\mathbf{k}$, *or*
2. ν is definably isomorphic to a type definable **subgroup** of $(\Gamma^n, +)$, *or* a **definable** subgroup of $((K/\mathcal{O})^n, +)$.

Moreover, in all cases, ν has unbounded exponent.

Dp-minimal groups

Corollary

If G as above, and $\text{dp-rk}(G) = 1$ then G is abelian-by-finite.

Proof

- By Simon, G contains a definable abelian normal $H \subseteq G$ such that G/H has bounded exponent.
- By Theorem, if G/H is infinite then it contains a subgroup of unbounded exponent, hence G/H must be finite. □

Simonetta's counter-example

There is a dp-minimal group in $\text{ACVF}_{p,p}$ that is solvable of step 2 (so not abelian-by-finite).

Previous (partial?) results, on groups and fields in valued fields

- Pillay on definable groups and fields in $\mathbb{Q}_p$,
- Hrushovski-Pillay on definable groups in local fields,
- Hrushovski-Rideau-Kikuchi on metastable groups and interpretable fields in ACVF,
- Bays-Pe, on definable fields in RCVF
- Acosta on 1-dim definable groups in $\mathbb{Q}_p$,
- Johnson-Yao on (non) definably compact groups in $\mathbb{Q}_p$,
- Johnson on interpretable groups in $\mathbb{Q}_p$,
- Onshuus-Vacaria groups in Presburger arithmetic.
- Gismatulin, Halupczock and Macpherson on definably simple groups in valued fields (in preperation).
- Cassani, on interpretability of trees in $\mathbb{Q}_p$ (in preparation)
- Alouf, Fornasiero, Gonzales, interpretable fields of dimension > 0 in $\mathbb{Q}_p$ (in preparation)

Back to the Theorem: Strong internality to a sort D

We have an interpretable group G in $\mathcal{K} \models$ RCVF, ACVF or pCF.

Definition

A definable $X \subseteq G$ is called **strongly internal** to a set D if there is an injective $f : X \to D^n$. If X has maximal dp-rank as such, we call it **D-critical**.

If $f : X \to D^n$ is finite-to-one we call X **almost strongly internal** to D. And if it has maximal dp-rank as such, **and** minimal finite fibers then we call it **almost D-critical**.

Warning

If $X \subseteq G$ is D-critical it does **not** mean that it is also almost D-critical. (Example, at the end if there is time)

So far, we showed:

There exists an infinite definable $X \subseteq G$, which is almost strongly internal to $D = K, K/\mathcal{O}, \mathbf{k}$ or Γ.

One simple case

The strongly minimal case

Let $\mathcal{K}$ be an ACVF.
Assume that there exists an infinite subset of G that is almost strongly internal to the residue field $\mathbf{k}$. Then there exists a definable infinite normal subgroup $H \trianglelefteq G$ and a finite $H_0 \trianglelefteq H$ such that H/H_0 is definably isomoprhic to a $\mathbf{k}$-algebraic group.

The proof is some local version of Zil'ber's indecomposability theorem.

Example

$G = (K/\mathbf{m}, +)$. Then $G \supseteq H = \mathcal{O}/\mathbf{m} \cong \mathbf{k}$, and H is algebraic.

Note: The quotient G/H is again of rank 1, and $G/H \cong K/\mathcal{O}$.

The unstable case: $\neg\,(\mathcal{K} \models ACVF \text{ and } D = \mathbf{k})$

Assume that either $\mathcal{K} \models$ RCVF, or $\mathcal{K} \models$ pCF, or $\mathcal{K} \models$ ACVF, but $D \neq \mathbf{k}$.

Main Lemma (almost true)

Assume that $X, Y \subseteq G$ are A-definable, infinite and almost D-critical, and let $(a, b) \in X \times Y$ be generic, namely ($\text{dp-rk}(a, b/A) = \text{dp-rk}(X) + \text{dp-rk}(Y) = 2\text{dp-rk}(X)$).

Then, there are $X_1 \subseteq X$, $Y_1 \subseteq Y$, $(a, b) \in X_1 \times Y_1$, definable over $B \supseteq A$, such that $\text{dp-rk}(a, b/B) = \text{dp-rk}(a, b/A)$, such that

$$X_1 \cdot Y_1 \subseteq X \cdot b \text{ and } X_1 \cdot Y_1 \subseteq a \cdot Y.$$

Warning

In the above $\text{dp-rk}(X_1 \cdot Y_1) = \text{dp-rk}(X) = \text{dp-rk}(Y)$ but we are **not** claiming that $\text{dp-rk}(X \cdot Y) = \text{dp-rk}(X) = \text{dp-rk}(Y)$! This is false in general.

Example-in any of the settings

Let $\pi : K \to K/\mathcal{O}$, and $G = K \times K/\mathcal{O}$, $\mathrm{dp\text{-}rk}(G) = 2$.

$$X = \{(x, \pi(x)) \in G : x \in K\} \quad Y = \{(x, -\pi(x)) \in G : x \in K\}.$$

Each has rank 1, and is in local bijection with K, so X, Y are strongly internal to K. They are also of maximal rank in G as such, so K-critical in G.

For $a = (x, \pi(x)) \in X$ and $b = (y, -\pi(y)) \in Y$, take

$$X_1 = (x + O) \times \{\pi(x)\} \subseteq X \text{ and } \; Y_1 = (y + O) \times \{-\pi(y)\} \subseteq Y.$$

These are cosets of the same subgroup of G, so
$X_1 + Y_1 = (x + y + O) \times \pi(x) + \pi(y) \subseteq X + b$.
So, $\mathrm{dp\text{-}rk}(X_1 + Y_1) = 1$.
However, $X + Y = G$, so $\mathrm{dp\text{-}rk}(X + Y) = 2$.

Interpretable groups (cont)

From finite-to-one to injective

Assume that $X \subseteq G$ is almost D-critical, witnessed by $f : X \to D^n$ (so f has smallest possible fibers). Then there is a finite normal subgroup $H \subseteq G$, and $X' \subseteq X$ with $\mathrm{dp}\text{-}\mathrm{rk}(X') = \mathrm{dp}\text{-}\mathrm{rk}(X)$, such that $f : X' \to D^n$ factors through $\pi : G \to G/H$.

Proof (a-la Hrushovski's thesis). For $(a, b) \in X^2$ generic, get $X_1, Y_1 \subseteq X$ as in main lemma. Then $X_1 \cdot Y_1 \subseteq X \cdot b$. Let $(a_1, b_1) \in X_1 \times Y_1$ generic over (a, b).
Consider $x \mapsto (f(x), f(xb_1 b^{-1})) \in D^{2n}$. It cannot have smaller fibers that f, thus... (here $[x]_f = f^{-1}(f(x))$).

$$[a_1]_f \cdot b_1 = [a_1]_f \cdot [b_1]_f = a \cdot [b_1]_f.$$

Finish using the following fact on groups

If $A, B \subseteq G$ and $A \cdot b = A \cdot B = a \cdot B$ then A and B are right and left cosets of the same subgroup H.

The infinitesimal vicinity

After replacing G with G/H for H finite and normal

We assume now that there exists an infinite definable $X \subseteq G$ which can be injected into D^m, for some distinguished sort D, all over some A. We take it of maximal rank, namely D-critical. We fix $a \in X$ with $\text{dp-rk}(a/A) = \text{dp-rk}(X) =: n$.

Definition: The infinitesimal vicinity of a in X

We consider the **global** partial type:

$$\nu_X(a) = \{Y \subseteq X \text{ definable over } B \subseteq M : a \in Y,\ \text{dp-rk}(a/B) = n\}.$$

By the "filter base" result, it is consistent and $\text{dp-rk}(\nu_X(a)) = \text{dp-rk}(X)$.

- When D is o-minimal then $\nu_X(a)$ is logically equivalent to the intersection of all $\mathcal{K}$-definable open neighborhoods of a in X.
- When $D = \Gamma$ is a $\mathbb{Z}$-group, and $a \in X = n \cdot \Gamma$, then $\nu_X(a)$ is all $\mathcal{K}$-definable "long" intervals containing a in $n\Gamma$. Well, not really.

The main properties of $\nu_X(a)$

Using the Main Lemma, we have:

Theorem

For $X \subseteq G$, a D-critical set, and $a \in X$ generic.

1. The set $\nu_X(a)$ is a coset of a type def. subgroup $\nu_X(a)a^{-1} \subseteq G$.
2. This subgroup does not depend on choice of a and the D-critical X, denote it by $\nu_D \subseteq G$, **the type-definable subgroup of G associated to the sort D.**
3. The group $\nu_D \subseteq G$ is (clearly) definably isomorphic to a group that is type-definable in the sort D.

The linearity of $D = K/\mathcal{O}$ and $D = \Gamma$

Facts/Theorem

1. In Γ: Definable functions in Γ are generically affine: $f(x) = L(x-a) + f(a)$, for $L : \Gamma^n \to \Gamma$ additive. This follows from QE, either for ordered vector space or Presburger arithmetic.
2. (HHP) In $K/\mathcal{O}$, all settings: every definable function is generically affine (uses definable Skolem functions in K and 1-h-minimality).

Using "group-chunk" methods, a-la Marikova,

Corollary

If $D = \Gamma$ or $D = K/\mathcal{O}$ then $\nu_D \subseteq G$ is definably isomorphic to a type definable **subgroup of** $(D^n, +)$.
When $D = K/\mathcal{O}$ then we can replace ν by a **definable** subgroup of G.
In both cases, ν_D has unbounded exponent.

The case of $D = K$ and real closed $D = \mathbf{k}$

Definable functions in $K \models ACVF \vee \mathrm{T}-convex \vee pcF$ and in o-minimal $\mathbf{k}$ are generically differentiable with respect to K and $\mathbf{k}$, respectively. Again, using group Chunk a-la-Marikova,

Corollary

If $D = K$ or $\mathbf{k}$ as above, then ν_D can be endowed with the structure of a differentiable group w.r to K and $\mathbf{k}$.

Corollary

The group ν_D is torsion-free (so G has unbounded exponent)

Proof (thanks to Starchenko)

- The group is differentiable. Consider $M(x, y) = x \cdot y$.
- The Jacobian of M at (e, e) is $Jac(M)_{e,e}(u, v) = u + v$.
- $\Rightarrow Jac(x \mapsto x^n)_e(v) = nv \neq 0$ ($char\, K, char\, \mathbf{k} \neq 0$).
- Hence, $x^n \neq e$ for x sufficiently close to e.
- $\Rightarrow \nu_D$ is torsion-free. □

Thoughts and Open questions

The dp-minimal case- a warning

It easily follows from the main theorem that there is no finite-to-finite correspondence between infinite subsets of $K, K/\mathcal{O}.\Gamma, \mathbf{k}$. Thus, if G is dp-minimal then it can be in correspondence with exactly one of the fours sorts.

However, if $G \sim D$ and $H \subseteq G$ normal infinite then G/H might be in correspondence with a different D.

E.g. $G = K/\mathbf{m}$ and $\mathbf{k} = \mathcal{O}/\mathbf{m} \subseteq G$ but $G/\mathbf{k} \cong K/\mathcal{O}$.

Given an interpretable group G,

- Is the sum of $\mathrm{dp\text{-}rk}(\nu_D)$, for $D = K, K/\mathcal{O}, \mathbf{k}, \Gamma$, equal to $\mathrm{dp\text{-}rk}(G)$?
- Is $\dim(G) = \dim(\nu_D)$, for $D = K$ (here $\dim$ is the geometric dimension)?
- What is the interaction between the four different groups ν_D's?
- Can the results be extended to the general P-minimal setting, to $ACVF_p$?

Interpretable fields

Using similar methods, we proved earlier:

Theorem

Let $\mathcal{K}$ be either
(i) (V-minimal expansion of) $\mathrm{ACVF}_{(0,0)}$, or
(ii) power bounded, T-convex, or
(iii) P-minimal with generic differentiability.

If F is interpretable field in $\mathcal{K}$ then it is definably isomorphic to a finite extension of K or $\mathbf{k}$.