Locally compact models for approximate rings

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In a breakthrough paper, Hrushovski proved the existence of the so-called locally compact (and in consequence also of Lie) models for pseudofinite approximate subgroups. This paved the way for Breuillard, Green, and Tao to give a full classification of all finite approximate subgroups. In fact, Hrushovski proved the existence of locally compact models in a much wider context. However, such models need not exist for arbitrary approximate subgroups; a counter-example can be found in my joint paper with Hrushovski and Pillay (refuting Wagner's conjecture). During the talk, I will focus on the existence of locally compact models for approximate subrings, where by an approximate subring of a ring we mean an additively symmetric subset X such that $X \cdot X \cup (X + X)$ is covered by finitely many additive translates of X. We prove that each approximate subring X of a ring has a locally compact model, i.e. a ring homomorphism $f : \langle X \rangle \longrightarrow S$ for some locally compact ring S such that $f[X]$ is relatively compact in S and there is a neighborhood U of 0 in S with $f^{-1}[U] \subseteq 4X + X \cdot 4X$ (where $4X := X + X + X + X$). This S is obtained as the quotient of the ring $\langle X \rangle$ interpreted in a suficiently saturated model by its type-definable ring connected component. The main point is to prove that this component always exists. More precisely, let X be a definable (in a structure M) approximate subring of a ring and $R := \langle X \rangle$. Let $\overline{X}$ be the interpretation of X in a sufficiently saturated elementary extension and $\overline{R} := \langle \overline{X} \rangle$. It follows from a paper by Massicot and Wagner that there exists the smallest M-type-definable subgroup of $(\overline{R}, +)$ of bounded index, which we denote by $(\overline{R}, +)^{00}_M$. We prove that $(\overline{R}, +)^{00}_M + \overline{R} \cdot (\overline{R}, +)^{00}_M$ is the smallest M-type-definable two-sided ideal of $\overline{R}$ of bounded index, which we denote by $\overline{R}^{00}_M$. Then the above S is just

$\bar{R}/R^{00}_M$ and $f : R \longrightarrow \bar{R}/R^{00}_M$ is the quotient map. In fact, f is the universal "definable" (in a suitable sense) locally compact model.