A LOWER DENSITY OPERATOR FOR THE BOREL ALGEBRA AND THE IDEAL OF COUNTABLE SETS

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Given a σ-algebra $S \subseteq \mathcal{P}(X)$ and a σ-ideal $J \subseteq S$, several authors have studied lower density operators and liftings $\Phi\colon S \to S$ with respect to J. Under **CH**, J. Von Neumann and M. H. Stone [3] proved the existence of a lifting for a Borel measure on $[0, 1]$. A simple proof of this fact was given later by Kazimierz Musiał [2].

We focus on Borel liftings for the σ-algebra $B(X)$ of Borel sets in an uncountable Polish space and for the ideal J of countable subsets of X. We show that the existence of a lifting on $B(X)$ with respect to J is equivalent to **CH**. Only the necessity is a new result. In particular, **CH** is equivalent to the existence of a lower density on $B(X)$ with respect to J which solves the problem of Jacek Hejduk posed in 2016.

We also derive an abstract theorem on the non-existence of a lifting for special algebras and ideals of subsets of $Y \times Y$ (where Y is of infinite cardinality) assuming a part of $\neg$**GCH**.

A final observation concerns the question whether a lower density operator $\Phi\colon B(\mathbb{R}) \to B(\mathbb{R})$ (which exists under **CH**) can have a range of bounded Borel level. The answer is negative.

This material is contained in a joint article with Szymon Głąb [1].

References

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